

Twists of Mukai bundles and the geometry of the level 3 modular variety over $\overline{\mathcal{M}}_8$

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Abstract

For a curve C of genus 6 or 8 and a torsion bundle η of order ℓ we study the vanishing of the space of global sections of the twist $E_C \otimes \eta$ of the rank two Mukai bundle E_C of C . The bundle E_C was used in a well-known construction of Mukai which exhibits general canonical curves of low genus as sections of Grassmannians in the Plücker embedding.

Globalizing the vanishing condition, we obtain divisors on the moduli spaces $\overline{\mathcal{R}}_{6,\ell}$ and $\overline{\mathcal{R}}_{8,\ell}$ of pairs $[C, \eta]$. First we characterize these divisors by different conditions on linear series on the level curves, afterwards we calculate the divisor classes. As an application, we are able to prove that $\overline{\mathcal{R}}_{8,3}$ is of general type.

1 Introduction

It is a famous result of Mukai (see [Muk93]) that a general canonical genus 8 curve is a linear section of the 8-dimensional Grassmannian $G(2, 6)$ in \mathbb{P}^{14} . In a similar fashion, the general genus 6 curve is the complete intersection of a 4-dimensional quadric and the 6-dimensional Grassmannian $G(2, 5)$ in \mathbb{P}^9 .

In both cases, the maps from the curve C to the Grassmannian are induced by the global sections of an (up to isomorphism) uniquely determined stable rank 2 bundle E_C with canonical determinant, which we call the *Mukai bundle* of C . We have $h^0(C, E_C) = 5$ in genus 6 and $h^0(C, E_C) = 6$ in genus 8. Since the bundle E_C captures the geometry of C , it is a natural problem to study loci of curves where E_C shows non-generic behaviour. In particular, we are interested in divisorial conditions involving E_C on moduli spaces of curves.

We let ℓ be a prime number, C a general curve of genus $g = 6$ or $g = 8$ and $\eta \in \text{Pic}^0(C)[\ell]$ a line bundle of order ℓ . Then we can consider the twisted bundle $E_C \otimes \eta$ and in particular its space of global sections $H^0(C, E_C \otimes \eta)$. Since the slope of E_C is $g - 1$, we expect $H^0(C, E_C \otimes \eta) = 0$ and the locus

$$\{[C] \in \mathcal{M}_g \mid H^0(E_C \otimes \eta) \neq 0 \text{ for some } \eta \in \text{Pic}^0(C)[\ell] \setminus \{\mathcal{O}_C\}\}$$

to be a divisor in \mathcal{M}_g , the moduli space of curves. In fact it is more natural to study the question on the modular variety $\mathcal{R}_{g,\ell}$ parametrizing pairs $[C, \eta]$ of smooth curves

of genus g together with a nontrivial ℓ -torsion line bundle. These spaces have been constructed and compactified in [CEFS13]. On $\mathcal{R}_{g,\ell}$ we define the locus

$$\mathcal{B}_{g,\ell} = \{[C, \eta] \in \mathcal{R}_{g,\ell} \mid H^0(C, E_C \otimes \eta) \neq 0\}$$

which is of codimension at most one in $\mathcal{R}_{g,\ell}$ and expected to be a divisor. This we prove:

Theorem 1.1. *In both $g = 6$ and $g = 8$ and for every prime ℓ the locus $\mathcal{B}_{g,\ell}$ is a divisor in $\mathcal{R}_{g,\ell}$.*

One of the primary motivations to study pairs $[C, \eta]$ originates from the study of *Prym varieties*. Recall that, for $\ell = 2$, such a pair $[C, \eta] \in \mathcal{R}_{g,2}$ corresponds to an étale double cover $\pi: C' \rightarrow C$. The Prym variety $\text{Pr}(C, \eta)$ associated to π is an abelian variety of dimension $g - 1$ which we construct by considering the *Norm map*

$$\text{Nm}_\pi: \text{Pic}^{2g-2}(C') \rightarrow \text{Pic}^{2g-2}(C), \quad \mathcal{O}_{C'}(D) \mapsto \mathcal{O}_C(\pi_* D)$$

and then letting

$$\text{Pr}(C, \eta) = \text{Nm}_\pi^{-1}(K_C)^+ = \{L \in \text{Nm}_\pi^{-1}(K_C) \mid h^0(C, L) \equiv 0 \pmod{2}\}$$

One can show that $\text{Pr}(C, \eta)$ is principally polarized. We then get a morphism

$$\mathcal{R}_{g,2} \rightarrow \mathcal{A}_{g-1}, \quad [C, \eta] \mapsto \text{Pr}(C, \eta)$$

called the *Prym map*, to the moduli space \mathcal{A}_{g-1} of principally polarized abelian varieties of dimension $g - 1$. Prym varieties play an important role in the study of \mathcal{A}_g since a general abelian variety of dimension at most 5 is a Prym. On the other hand, recall that the general abelian variety of dimension at least 4 is not the Jacobian of a curve. Hence Prym varieties make the study of abelian varieties amenable to techniques from curve theory in a larger range than by just studying Jacobians.

For $\ell \geq 3$ one can analogously assign a cyclic unramified cover $C' \rightarrow C$ of degree ℓ to any pair $[C, \eta] \in \mathcal{R}_{g,\ell}$. However, this process is only reversible if we consider such covers together with a generator of their Galois group.

Our main interest lies in furthering the understanding of the birational geometry of the spaces $\mathcal{R}_{g,\ell}$. It is known (see [FL10] and [Bru16]) that $\mathcal{R}_{g,2}$ is of general type for $g \geq 14$ while $\mathcal{R}_{g,3}$ is known to be of general type for $g \geq 12$ (see [CEFS13]). We can use the divisor $\mathcal{B}_{8,3}$ previously constructed to prove the following:

Theorem 1.2. *$\mathcal{R}_{8,3}$ is of general type.*

Note that we now have a result for genus 8 and level 3 while there is currently nothing known about $\mathcal{R}_{9,3}$ and $\mathcal{R}_{10,3}$. The Kodaira dimension of $\mathcal{R}_{11,3}$ is at least 19 (proved in [CEFS13]) but our theorem actually suggests that all three spaces should be of general type.

We recall that Mukai bundles also exist in genus 7 and 9. Hence one can hope to make use of them in order to exhibit divisors similar to $\mathcal{B}_{8,3}$. The difficulty is that in genus 7 the Mukai bundle E_C has rank 5 and in genus 9 it is of rank 3, making its slope a

non-integer rational number in both cases. An approach using vanishing conditions of global sections of a twist $E_C \otimes \eta$ is therefore not going to work. Possibly one can use, e.g. for $g = 9$, the bundle $\text{Sym}^3 E_C$ which has integral slope $2g - 2$.

We also note that, although Theorem 1.1 includes genus 6, the resulting divisor $\mathcal{B}_{6,3}$ does not enable us to prove a statement similar to Theorem 1.2. On the other hand, $\mathcal{R}_{g,3}$ is known to be unirational for $g \leq 5$ ([BC10; BV10; Ver16]). This makes the study of $\mathcal{R}_{6,3}$ especially interesting, since it is likely to be a transitional case.

We now explain the strategy of the proof of Theorem 1.2. The method of obtaining general type results is by constructing divisors with a divisor class in $\text{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,\ell})$ satisfying certain numerical bounds. Here $\overline{\mathcal{R}}_{g,\ell}$ is the modular compactification obtained by using quasi-stable level ℓ curves as described in [CEFS13]. Hence the first step is to calculate the divisor class of the closure $\overline{\mathcal{B}}_{g,\ell}$ in the compactification $\overline{\mathcal{R}}_{g,\ell}$. We do this for both $g = 6$ and 8 and for all ℓ .

In fact it is enough to calculate the class on an appropriate partial compactification $\mathcal{R}'_{g,\ell}$ of $\mathcal{R}_{g,\ell}$ containing only smooth and irreducible one-nodal curves. On a cover of $\mathcal{R}'_{g,\ell}$ we express the closure of $\mathcal{B}_{g,\ell}$ as the degeneracy locus of a morphism $\phi_{g,\ell}$ between vector bundles of the same rank. Using Porteous' formula and the machinery for calculating Chern classes of vector bundles over $\overline{\mathcal{M}}_g$, developed in [Far09], we then show:

Theorem 1.3. *We have the following expressions for the pushforward to $\mathcal{R}'_{g,\ell}$ of the classes of the degeneracy loci of $\phi_{g,\ell}$:*

a) *The virtual class of the closure of $\mathcal{B}_{6,\ell}$ in $\mathcal{R}'_{6,\ell}$ is given by*

$$[\overline{\mathcal{B}}_{6,\ell}]^{\text{virt}} = 35\lambda - 5(\delta'_0 + \delta''_0) - \frac{5}{\ell} \sum_{a=1}^{\lfloor \ell/2 \rfloor} (\ell^2 - a\ell + a^2) \delta_0^{(a)}$$

b) *The virtual class of the closure of $\mathcal{B}_{8,\ell}$ in $\mathcal{R}'_{8,\ell}$ is given by*

$$[\overline{\mathcal{B}}_{8,\ell}]^{\text{virt}} = 196\lambda - 28(\delta'_0 + \delta''_0) - \frac{14}{\ell} \sum_{a=1}^{\lfloor \ell/2 \rfloor} (2\ell^2 - a\ell + a^2) \delta_0^{(a)}$$

In particular, the classes $[\overline{\mathcal{B}}_{g,\ell}]^{\text{virt}} - n[\overline{\mathcal{B}}_{g,\ell}]$ are effective and entirely supported on the boundary of $\mathcal{R}'_{g,\ell}$ for some $n \geq 1$.

By describing the degeneracy of the morphism used in Porteous' formula along the boundary we can improve these divisor classes still further. Similarly, we also improve a divisor class found in [CEFS13]. Combining these results we can prove our Main Theorem 1.2.

Outline of the paper We begin in section 2 by recalling some facts about the moduli spaces $\overline{\mathcal{R}}_{g,\ell}$. This is followed in section 3 by a review of the results we need about Mukai bundles. Then, in section 4, we discuss the loci $\mathcal{B}_{g,\ell}$ and show that they are divisors. To

do this, we prove a more general statement about vanishing of global sections of twists of semistable vector bundles under the right hypotheses (see Theorem 4.1). We also reinterpret the vanishing condition in terms of injectivity of certain maps of linear series. Afterwards we proceed to calculate the class of the divisors in section 5. The degeneracy calculation and the application to $\overline{\mathcal{R}}_{8,3}$ can be found in section 6.

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2 Moduli spaces of quasi-stable level ℓ curves

Let ℓ be a prime number. By $\mathcal{R}_{g,\ell}$ we denote the moduli space of isomorphism classes of pairs $[C, \eta]$ where C is a curve of genus g and η is a line bundle of order ℓ on C . In this section we describe the basic facts about these moduli spaces and their compactifications by quasi-stable level ℓ curves.

2.1 Modular compactification

Several constructions to compactify $\mathcal{R}_{g,\ell}$ have been put forward. Initially the theory was focused on the case $\ell = 2$ of Prym curves, for which A. Beauville put forward the theory of admissible covers ([Bea77] and [ACV03]). It extends the modular description of points in $\mathcal{R}_{g,2}$ as étale double covers $C' \rightarrow C$ to stable curves. Later, M. Bernstein in her PhD thesis [Ber99] considered the normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_{g,\ell}$. Closed points of the ensuing compactification $\overline{\mathcal{R}}_{g,\ell}$ correspond to stable curves with torsion line bundles on each component and, over irreducible nodes, additionally the ℓ -th roots of line bundles of the form $\mathcal{O}_{\tilde{C}}(ap + (\ell - a)q)$. Here p, q are the two points of the normalization lying over the node.

After the study of moduli spaces of roots of line bundles by L. Caporaso, C. Casagrande and M. Cornalba in [CCC07], it became clear what the right definition of limits of level $\ell \geq 3$ curves should be. The study of moduli spaces of these *quasi-stable level ℓ curves* was initiated in [CEFS13]. This very convenient modular interpretation for the geometric points of $\overline{\mathcal{R}}_{g,\ell}$ we are going to introduce here and use subsequently.

Definition 2.1. A *quasi-stable* curve of genus g is a connected nodal curve C of arithmetic genus g such that:

1. Every smooth rational component E of C meets the rest of C in at least two points, i.e., we have $n_E := |E \cap (\overline{C \setminus E})| \geq 2$.

2. If E and E' are two such components with $n_E = n_{E'} = 2$, then we have $E = E'$ or $E \cap E' = \emptyset$.

A smooth rational component E with $n_E = 2$ is called *exceptional*.

Note that by blowing down all exceptional components of a quasi-stable curve we obtain a stable curve.

Definition 2.2. A *quasi-stable level ℓ curve* of genus g is a triple $[C, \eta, \beta]$ consisting of a quasi-stable curve C of genus g , a line bundle $\eta \in \text{Pic}^0(C)$ and a sheaf homomorphism $\beta: \eta^{\otimes \ell} \rightarrow \mathcal{O}_C$, subject to the following conditions:

1. For each exceptional component E of C we have $\eta|_E = \mathcal{O}_E(1)$.
2. For each non-exceptional component the morphism β is an isomorphism.
3. For each exceptional component E and $\{p, q\} = E \cap \overline{C \setminus E}$ we have

$$\text{ord}_p(\beta) + \text{ord}_q(\beta) = \ell$$

A *family of quasi-stable level ℓ curves* over a scheme S is a triple $(\mathcal{C} \rightarrow S, \eta, \beta)$ where $\mathcal{C} \rightarrow S$ is a flat family of quasi-stable curves, η is a line bundle on \mathcal{C} and $\beta: \eta^{\otimes \ell} \rightarrow \mathcal{O}_{\mathcal{C}}$ is a sheaf homomorphism such that for each geometric fiber $C_s \rightarrow \{s\} \subset S$ the triple $(C_s, \eta|_{C_s}, \beta|_{C_s})$ is a quasi-stable level ℓ curve.

The fibered category of families of quasi-stable level ℓ curves defines a Deligne–Mumford stack whose associated coarse moduli space we denote by $\text{Root}_{g,\ell}$. Since for $\ell > 3$ the singularities of $\text{Root}_{g,\ell}$ are not normal, the definition of the actual moduli space $\overline{\mathcal{R}}_{g,\ell}$ is a bit more involved. It arises as a connected component of the coarse moduli space $\overline{\mathcal{M}}_g(\mathbb{B}\mathbb{Z}_\ell)$ of twisted level curves ([ACV03]), which is a normalization of $\text{Root}_{g,\ell}$. In particular the treatment of the universal curve over the Deligne–Mumford stack $\overline{\mathcal{R}}_{g,\ell}$ requires some further work. We direct the reader to the extensive discussions in [Chi08] and [CEFS13].

2.2 Boundary divisors

Let $\pi: \overline{\mathcal{R}}_{g,\ell} \rightarrow \overline{\mathcal{M}}_g$ be the forgetful map. We study the boundary components of $\overline{\mathcal{R}}_{g,\ell}$. They lie over the boundary of $\overline{\mathcal{M}}_g$, so we can examine the components lying over Δ_i for $i = 0, \dots, \lfloor \frac{g}{2} \rfloor$. Because of notational convenience sometimes boundary components of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{R}}_{g,\ell}$ will be denoted by the same symbols. However it should always be clear from the context which space we are considering.

The divisors $\Delta_i, \Delta_{g-i}, \Delta_{g:i}, i \geq 1$. First consider $i \geq 1$ and let $X \in \Delta_i$ be general, i.e., $X = C \cup D$ is the union of two curves of genera i and $g-i$ meeting transversally in a single node. The line bundle $\eta \in \text{Pic}^0(X)$ on the corresponding level ℓ curve is determined by its restrictions $\eta_C = \eta|_C$ and $\eta_D = \eta|_D$ satisfying $\eta_C^{\otimes \ell} = \mathcal{O}_C$ and $\eta_D^{\otimes \ell} = \mathcal{O}_D$.

Either one of η_C and η_D (but not both) can be trivial, so $\pi^*(\Delta_i)$ splits into three irreducible components Δ_i , Δ_{g-i} and $\Delta_{i:g-i}$ where the general element in Δ_i is $[C \cup D, \eta_C \neq \mathcal{O}_C, \mathcal{O}_D]$, the generic point of Δ_{g-i} is of the form $[C \cup D, \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$ and the generic point of $\Delta_{i:g-i}$ looks like $[C \cup D, \eta_C \neq \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$. Observe that for $i = 1$ and $\ell \geq 3$, due to the extra automorphism on elliptic tails, we have the pullback formula $\pi^*(\Delta_1) = 2\Delta_1 + 2\Delta_{1:g-1} + \Delta_{g-1}$ and the map π is ramified along Δ_1 and $\Delta_{1:g-1}$.

The divisor Δ_0'' . Now let $i = 0$. The generic point of Δ_0 in $\overline{\mathcal{M}}_g$ is a one-nodal irreducible curve C of geometric genus $g - 1$. We first consider points of the form $[C, \eta]$ lying over C , i.e., without an exceptional component. Denote by $\nu: \tilde{C} \rightarrow C$ the normalization and by p, q the preimages of the node. Then we have an exact sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow \text{Pic}^0(C) \xrightarrow{\nu^*} \text{Pic}^0(\tilde{C}) \rightarrow 0$$

which restricts to

$$0 \rightarrow \mathbb{Z}/\ell\mathbb{Z} \rightarrow \text{Pic}^0(C)[\ell] \xrightarrow{\nu^*} \text{Pic}^0(\tilde{C})[\ell] \rightarrow 0$$

on the ℓ -torsion part. The group $\mathbb{Z}/\ell\mathbb{Z}$ represents the ℓ possible choices of gluing the fibers at p and q for each line bundle in $\text{Pic}^0(\tilde{C})[\ell]$. For the case $\nu^*\eta = \mathcal{O}_{\tilde{C}}$ there are exactly $\ell - 1$ possible choices of $\eta \neq \mathcal{O}_C$. These curves $[C, \eta]$ correspond to the order ℓ analogues of the classical *Wirtinger double covers*

$$\tilde{C}_1 \amalg \tilde{C}_2 / (p_1 \sim q_2, p_2 \sim q_1) \xrightarrow{2:1} \tilde{C} / (p \sim q) = C$$

We denote by Δ_0'' the closure of the locus of level ℓ Wirtinger covers. Note that for $\ell > 3$ the divisor Δ_0'' is not irreducible. Indeed, up to switching the role of the points p and q lying over the node, the sections s of an ℓ -torsion line bundle $\eta' \in \text{Pic}^0(\tilde{C})$ that descend to C are determined by $s(p) = \xi^a s(q)$ where ξ is an ℓ -th root of unity and $1 \leq a \leq \ell - 1$. Hence we get precisely $\lfloor \ell/2 \rfloor$ irreducible components and each of them has order 2 over $\Delta_0 \subset \overline{\mathcal{M}}_g$.

The divisor Δ_0' . On the other hand, there are $\ell^{2(g-1)} - 1$ nontrivial elements in the group $\text{Pic}^0(\tilde{C})[\ell]$. For each of them there are ℓ choices of gluing, so we have a total of $\ell \cdot (\ell^{2g-2} - 1)$ choices for $\eta \in \text{Pic}^0(C)$ such that $\nu^*\eta \neq \mathcal{O}_{\tilde{C}}$. We let Δ_0' be the closure of the locus of pairs $[C, \eta]$ such that $\nu^*\eta \neq \mathcal{O}_{\tilde{C}}$.

The divisors $\Delta_0^{(a)}$. We turn to the case of curves of the form $[X = \tilde{C} \cup_{p,q} E, \eta]$ where E is an exceptional component. The stabilization of such a curve is again a one-nodal curve C . Denote by β the morphism $\eta^{\otimes \ell} \rightarrow \mathcal{O}_X$. Since $\eta|_E = \mathcal{O}_E(1)$, we must have $\beta_{E \setminus \{p,q\}} = 0$ and $\deg(\eta^{\otimes \ell}|_{\tilde{C}}) = -\ell$. By swapping p and q if necessary, we can conclude that $\eta^{\otimes \ell}|_{\tilde{C}} = \mathcal{O}_{\tilde{C}}(-ap - (\ell - a)q)$ for some integer a with $1 \leq a \leq \lfloor \ell/2 \rfloor$. There are $\ell^{2(g-1)}$ choices of square roots of $\mathcal{O}_{\tilde{C}}(-ap - (\ell - a)q)$ and each of these determines uniquely a Prym curve $[X, \eta]$ of this form. We denote the closure of the locus of such curves by $\Delta_0^{(a)}$. Then the degree of $\Delta_0^{(a)}$ over Δ_0 is $2\ell^{2g-2}$ for all a . The factor 2 arises because of the symmetry in p and q .

2.3 The canonical class

Our goal is to show that $\overline{\mathcal{R}}_{8,3}$ is of general type, i.e., we have to show that the canonical class $K_{\widehat{\mathcal{R}}_{8,3}}$ is big for some desingularization $\widehat{\mathcal{R}}_{8,3}$ of $\overline{\mathcal{R}}_{8,3}$. An extension result for pluricanonical forms (Remark 3.5 in [CEFS13]) shows that in fact we do not need to pass to a desingularization but can perform all calculations on $\mathcal{R}_{8,3}$ directly.

Let us denote by δ_i, δ'_0 etc. the rational divisor classes associated to the respective boundary divisors. The canonical class of $\overline{\mathcal{R}}_{g,\ell}$ for $g \geq 4$ and $\ell \geq 3$ then has the following expression (see [CEFS13], Proposition 1.5):

$$K_{\overline{\mathcal{R}}_{g,\ell}} = 13\lambda - 2(\delta'_0 + \delta''_0) - (\ell + 1) \sum_{k=1}^{\lfloor \ell/2 \rfloor} \delta_0^{(k)} - 2 \sum_{i=1}^{\lfloor g/2 \rfloor} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - \delta_{g-1}$$

Hence, if we can find an effective divisor

$$E \equiv \alpha\lambda - b'_0\delta'_0 - b''_0\delta''_0 - \sum_{k=1}^{\lfloor \ell/2 \rfloor} b_0^{(k)}\delta_0^{(k)} - \sum_{i=1}^{\lfloor g/2 \rfloor} (b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i})$$

such that

$$\frac{\alpha}{b'_0}, \frac{\alpha}{b''_0}, \frac{\alpha}{b_i}, \frac{\alpha}{b_{g-i}}, \frac{\alpha}{b_{i:g-i}} < \frac{13}{2}, \quad \frac{\alpha}{b_{g-1}} < \frac{13}{3}, \quad \frac{\alpha}{b_0^{(k)}} < \frac{13}{\ell+1} \quad (2.1)$$

then it follows that we can write

$$K_{\overline{\mathcal{R}}_{g,\ell}} \equiv \varepsilon\lambda + \alpha E + \beta D$$

where D is supported on the boundary, $\alpha, \beta \geq 0$ and $\varepsilon > 0$. Since the Hodge class λ is big, $K_{\overline{\mathcal{R}}_{g,\ell}}$ must be big as well.

Importantly, at least for $g \leq 23$, the only relevant data are the coefficients of $\lambda, \delta'_0, \delta''_0$ and $\delta_0^{(k)}$:

Lemma 2.3 ([CEFS13], Remark 3.5). *Let $g \leq 23$ and $\ell \geq 2$. In order to prove that $K_{\overline{\mathcal{R}}_{g,\ell}}$ is big it is enough to exhibit an effective divisor*

$$E \equiv \alpha\lambda - b'_0\delta'_0 - b''_0\delta''_0 - \sum_{k=1}^{\lfloor \ell/2 \rfloor} b_0^{(k)}\delta_0^{(k)} - \sum_{i=1}^{\lfloor g/2 \rfloor} (b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i})$$

with $\alpha/b'_0 < 13/2$, $\alpha/b''_0 < 13/2$ and $\alpha/b_0^{(k)} < 13/(\ell+1)$ for all $k = 1, \dots, \lfloor \ell/2 \rfloor$. The coefficients b_i, b_{g-i} and $b_{i:g-i}$ are then automatically suitably bounded.

3 Mukai bundles

Let $g = 6$ or $g = 8$. We will always denote the Mukai bundle associated to a curve C by E_C . By the results of [Muk93] it is possible to give explicit Brill–Noether type conditions for a curve to arise as a complete intersection with a Grassmannian:

Theorem 3.1 ([Muk93], Main Theorem). *A curve C of genus 8 is a transversal linear section of the 8-dimensional Grassmannian $G(2, 6) \subseteq \mathbb{P}^{14}$ if and only if C has no \mathfrak{g}_7^2 .*

Theorem 3.2 ([Muk93], §5). *A curve C of genus 6 is the complete intersection of $G(2, 5)$ and a 4-dimensional quadric in \mathbb{P}^9 if and only if $W_4^1(C)$ is finite, i.e., C is not trigonal, not a plane quintic and not bielliptic.*

We get the vector bundles E_C in question by restricting the tautological bundle of the Grassmannian to C . Importantly for us, it turns out that the existence of a vector bundle with the right numerics is guaranteed by slightly weaker assumptions:

Theorem 3.3 ([Muk93], §5). *Let C be a curve of genus 6 which is neither trigonal nor a plane quintic. When F runs over all stable rank 2 bundles with canonical determinant on C , the maximum of $h^0(C, F)$ is equal to 5. Moreover, such vector bundles E_C on C with $h^0(C, E_C) = 5$ are unique up to isomorphism and generated by global sections.*

Theorem 3.4 ([Muk93], §3). *Let C be a curve of genus 8 without a \mathfrak{g}_4^1 . When F runs over all semistable rank 2 bundles with canonical determinant on C , the maximum of $h^0(C, F)$ is equal to 6. Moreover, such vector bundles E_C on C with $h^0(C, E_C) = 6$ are unique up to isomorphism and generated by global sections.*

We denote the locus of curves satisfying the assumptions of Theorem 3.3 or 3.4 by \mathcal{M}_g^μ and we set $\mathcal{R}_{g,\ell}^\mu = \mathcal{M}_g^\mu \times_{\mathcal{M}_g} \mathcal{R}_{g,\ell}$. The codimension of the complement of this locus is two: In genus 6 the trigonal locus has codimension 2 and the locus of plane quintics has codimension 3. In genus 8, the tetragonal locus is also of codimension 2.

We remark in passing that on genus 7 and genus 9 curves there also exist special Mukai bundles. They have rank 5 and 3, respectively. Analogously to their counterparts in genus 6 and 8 their global sections give embeddings of the curve, albeit in an orthogonal or a symplectic Grassmannian. These bundles too exhibit interesting properties: for instance on a general genus 9 curve they were used to give early counterexamples to Mercat's conjecture (see [LMN12]).

There is a more explicit construction of the bundles E_C in question:

Lemma 3.5. *Let C be a curve of genus 6, not trigonal and not a plane quintic, and $A \in W_4^1(C)$. Set $L = \omega_C \otimes A^{-1}$. The bundle E_C is given as the unique nontrivial extension of L by A with a 5-dimensional space of global sections.*

We quickly describe the main steps of the proof because they will be useful in section 4. The full details can be found in [Muk93]. Consider any extension

$$0 \rightarrow A \rightarrow F \rightarrow L \rightarrow 0$$

and the resulting exact sequence in cohomology:

$$0 \rightarrow H^0(C, A) \rightarrow H^0(C, F) \rightarrow H^0(C, L) \xrightarrow{\delta_F} H^1(C, A) \rightarrow \dots$$

Then $h^0(C, F) \leq h^0(C, A) + h^0(C, L) = 5$ with equality if and only if $\delta_F = 0$. By Serre duality we have

$$\mathrm{Ext}^1(L, A) \cong H^1(C, A \otimes L^{-1}) \cong H^0(C, L^{\otimes 2})^\vee$$

while the boundary homomorphism δ_F lies in

$$\mathrm{Hom}(H^0(C, L), H^1(C, A)) = \mathrm{Hom}(H^0(C, L), H^0(C, L)^\vee) = H^0(C, L)^\vee \otimes H^0(C, L)^\vee$$

We have a map

$$\mathrm{Ext}^1(L, A) = H^0(C, L^{\otimes 2})^\vee \rightarrow H^0(C, L)^\vee \otimes H^0(C, L)^\vee, \quad (3.1)$$

given by $F \mapsto \delta_F$, which is dual to the multiplication map of sections

$$H^0(C, L) \otimes H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$$

It turns out that the cokernel of this map is $H^0(C, L^{\otimes 2})/\mathrm{Sym}^2 H^0(C, L)$, which is 1-dimensional. Hence, up to scaling there is a unique nonzero element in the kernel of the map (3.1). It can be checked that this construction does not depend on the choice of A .

In a completely analogous fashion, we get the following result in genus 8:

Lemma 3.6. *Let C be a curve of genus 8 with $W_4^1(C) = \emptyset$ and $A \in W_5^1(C)$. Set $L = \omega_C \otimes A^{-1}$. The bundle E_C is given as the unique nontrivial extension of L by A with a 6-dimensional space of global sections.*

4 Constructing the divisors

In both genera, the slope of E_C is

$$\mu(E_C) = \frac{\deg \det E_C}{\mathrm{rk} E_C} = \frac{2g-2}{2} = g-1$$

hence $\chi(E_C) = 0$. We can therefore consider the virtual theta divisor of E_C

$$\Theta_{E_C} = \{\xi \in \mathrm{Pic}^0(C) \mid H^0(C, E_C \otimes \xi) \neq 0\}$$

in $\mathrm{Pic}^0(C)$. Since E_C is a semistable rank 2 vector bundle, Θ_{E_C} is indeed of codimension one (see [Ray82], Proposition 1.6.2). By intersecting $\mathcal{R}_{g,\ell}$ and the locus $\{[C, \xi] \mid \xi \in \Theta_{E_C}\}$ in the universal Jacobian, we obtain that

$$\mathcal{B}_{g,\ell} = \{[C, \eta] \in \mathcal{R}_{g,\ell} \mid H^0(C, E_C \otimes \eta) \neq 0\}$$

is of codimension at most one in $\mathcal{R}_{g,\ell}$ and expected to be a divisor.

4.1 Transversality

We will show that on the general pair $[C, \eta] \in \mathcal{R}_{g,\ell}$ we have $H^0(C, E_C \otimes \eta) = 0$. More precisely, we will show that on the general smooth curve C there exists an ℓ -torsion bundle η such that $H^0(C, E_C \otimes \eta) = 0$.

Let C be a smooth and connected curve of genus g and let E be a semistable vector bundle of rank $r \geq 2$ and determinant $\det(E) = \vartheta^{\otimes r}$, where ϑ is a theta characteristic. The *theta divisor* of ϑ is

$$\Theta = \{\xi \in \text{Pic}^0(C) \mid H^0(C, \vartheta \otimes \xi) \neq 0\}$$

Assume further that $h^0(C, E) \geq 1$ and that E admits a theta-divisor, i.e., the set

$$\Theta_E^{\text{set}} = \{\xi \in \text{Pic}^0(C) \mid H^0(C, E \otimes \xi) \neq 0\}$$

is a proper subset of $\text{Pic}^0(C)$. Then Θ_E^{set} is the support of a natural divisor $\Theta_E \in |r\Theta|$, called a *theta divisor* of E (see for instance [Bea04]). We now consider the vector bundle $F := E \oplus \vartheta^{\oplus(\ell-r)}$. As a direct sum of semistable vector bundles of the same slope it is itself semistable. Furthermore, it admits an associated theta divisor Θ_F by assumption on E . Then

$$\text{Supp } \Theta_F = \text{Supp } \Theta_E \cup \text{Supp } \Theta$$

and we have

$$\Theta_F = \Theta_E + (\ell - r)\Theta \in |\ell\Theta|$$

We now let $J = \text{Pic}^0(C)$ be the Jacobian of C and $f_\ell: J \rightarrow \mathbb{P}H^0(J, \mathcal{O}_J(\ell\Theta))^* = \mathbb{P}^{\ell^g-1}$ be the morphism defined by the linear system $|\ell\Theta|$. Recall from [Mum66] that the representation of the theta group of $\mathcal{O}_J(\ell\Theta)$ is irreducible. Any linear subspace of \mathbb{P}^{ℓ^g-1} containing the image $f_\ell(J[\ell])$ of the set of ℓ -torsion points of J corresponds to an invariant subspace of the representation of the theta group, hence $f_\ell(J[\ell])$ is not contained in a hyperplane.

Summarizing this discussion, we have:

Theorem 4.1. *Let C be a smooth connected curve. Let E be a semistable vector bundle on C of rank $r \geq 2$ and determinant $\det(E) = \vartheta^{\otimes r}$ with ϑ a theta characteristic. Assume $H^0(C, E) \neq 0$ and E admits a theta divisor. Then there exists a non-trivial $\eta \in \text{Pic}^0(C)[\ell]$ with $H^0(C, E \otimes \eta) = 0$.*

Proof. Consider as before $F = E \oplus L^{\oplus(\ell-r)}$. Then, by the above analysis, there is some torsion point $\eta \in \text{Pic}^0(C)[\ell]$ which is not contained in the hyperplane of \mathbb{P}^{ℓ^g-1} defined by $\Theta_F = \Theta_E + (\ell - r)\Theta$. Hence η is not contained in Θ_E , which means $H^0(C, E \otimes \eta) = 0$. By assumption $H^0(C, E \otimes \mathcal{O}_C) \neq 0$ and therefore $\eta \neq \mathcal{O}_C$. ■

In particular, we can apply the above theorem to E_C , which is stable of rank 2 and has canonical determinant, hence satisfies all the assumptions. We conclude that $\mathcal{B}_{g,\ell}$ is a divisor for every ℓ and both genera $g = 6$ and $g = 8$.

4.2 Reinterpreting the divisor

To calculate the divisor classes of $\mathcal{B}_{g,\ell}$, it will be necessary first to give other characterizations of the pairs $[C, \eta] \in \mathcal{B}_{g,\ell}$.

In the following discussion we will only consider $[C, \eta] \in \mathcal{R}_{g,\ell}^\mu$. As discussed in Lemma 3.5, the bundle E_C is an extension

$$0 \rightarrow A \rightarrow E_C \rightarrow \omega_C \otimes A^{-1} \rightarrow 0$$

where $A \in W_4^1(C)$ if $g = 6$ and $A \in W_5^1(C)$ if $g = 8$. After tensoring with η , the associated long exact sequence in cohomology starts with

$$0 \rightarrow H^0(C, A \otimes \eta) \rightarrow H^0(C, E_C \otimes \eta) \rightarrow H^0(C, \omega_C \otimes A^{-1} \otimes \eta) \xrightarrow{\delta_{E_C \otimes \eta}} H^1(C, A \otimes \eta)$$

We immediately get:

Lemma 4.2. $[C, \eta] \in \mathcal{B}_{g,\ell}$ if and only if there exists an A such that $H^0(C, A \otimes \eta) \neq 0$ or the boundary map

$$\delta_{E_C \otimes \eta}: H^0(C, \omega_C \otimes A^{-1} \otimes \eta) \rightarrow H^1(C, A \otimes \eta) \quad (4.1)$$

is not an isomorphism.

Since $H^0(C, A \otimes \eta) \neq 0$ happens only on curves in a subvariety of codimension at least 2 in $\mathcal{R}_{g,\ell}$, in what follows we will ignore the locus of such curves. This does not affect divisor class calculations.

In genus 6, we can give another interpretation of $\mathcal{B}_{6,\ell}$. Let $A \in W_4^1(C)$ and $L = \omega_C \otimes A^{-1}$. By Riemann–Roch, we have $h^0(C, L \otimes \eta) = 1$ and also $h^1(C, A \otimes \eta) = 1$. So for (4.1) to be an isomorphism it is enough for it to be nonzero.

Lemma 4.3. In the case $g = 6$, the boundary map $\delta_{E_C}: H^0(C, L \otimes \eta) \rightarrow H^1(C, A \otimes \eta)$ is nonzero if and only if the multiplication map followed by projection

$$H^0(C, L \otimes \eta) \otimes H^0(C, L \otimes \eta^{-1}) \xrightarrow{m_\eta} H^0(C, L^{\otimes 2}) \xrightarrow{p} H^0(C, L^{\otimes 2}) / \text{Sym}^2 H^0(C, L) \quad (4.2)$$

is an isomorphism.

Proof. Since C is not a plane quintic, L is base point free, so it induces a morphism to \mathbb{P}^2 . The image is birational to C if and only if C is not trigonal and not bielliptic, so for a general genus 6 curve L induces a birational map to a 4-nodal plane sextic. This implies that the multiplication map $\text{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$ is injective. So both domain and codomain of the map $p \circ m_\eta$ are 1-dimensional. For a bielliptic curve $C \rightarrow E$ the same conclusion holds by $H^0(C, L) \cong H^0(E, \mathcal{O}_E(1))$. Hence (4.2) is an isomorphism if and only if it is nonzero.

Extensions of L by A and of $L \otimes \eta$ by $A \otimes \eta$ are both parametrized by

$$\text{Ext}^1(L, A) \cong \text{Ext}^1(L \otimes \eta, A \otimes \eta) \cong H^1(C, A \otimes L^{-1}) \cong H^0(C, L^{\otimes 2})^\vee$$

while the boundary morphism $\delta_{E_C \otimes \eta}$ lives in

$$\begin{aligned} \text{Hom}(H^0(C, L \otimes \eta), H^1(C, A \otimes \eta)) &\cong H^0(C, L \otimes \eta)^\vee \otimes H^1(C, A \otimes \eta) \\ &\cong H^0(C, L \otimes \eta)^\vee \otimes H^0(C, L \otimes \eta^{-1})^\vee \end{aligned}$$

and we have a map

$$\alpha: H^0(C, L^{\otimes 2})^\vee \rightarrow H^0(C, L \otimes \eta)^\vee \otimes H^0(C, L \otimes \eta^{-1})^\vee \quad (4.3)$$

sending an extension $E \otimes \eta$ to the boundary homomorphism $\delta_{E \otimes \eta}$. Note that α is the dual of the multiplication map m_η . We denote by $[\alpha]$ the composition of α with the dual of the projection p .

The space $H^0(C, L^{\otimes 2})^\vee / \text{Sym}^2 H^0(C, L)^\vee$ is generated by the class $[\phi_{E_C}]$ of the map corresponding to the Mukai bundle E_C (see the discussion after Lemma 3.5). Now (4.2) is the zero map if and only if the dual map $[\alpha]$ is the zero map if and only if $[\phi_{E_C}]$ is mapped to 0 by $[\alpha]$, i.e., if $[\phi_{E_C}] \circ (p \circ m_\eta) = 0$. But this is exactly the boundary map $\delta_{E_C \otimes \eta}$ given by the image of the extension $E_C \otimes \eta$ under (4.3). ■

Remark 4.4. For the case $\ell = 2$ further descriptions of the divisor exist. A general curve $[C, \eta] \in \mathcal{B}_{6,2}$ equivalently satisfies the following conditions:

- a) C has a 4-nodal plane sextic model with a totally tangent conic, i.e., there exists an $L \in W_6^2(C)$ inducing a birational map to $\Gamma \subseteq \mathbb{P}^2$, and a conic $Q \subseteq \mathbb{P}^2$ with $Q \cap \Gamma = 2D$ for some $D \in C^{(6)}$. This identification follows from Lemma 4.3.
- b) The Prym map $\mathcal{R}_{6,2} \rightarrow \mathcal{A}_5$ is ramified at $[C, \eta]$ (see [FGSMV14], Theorem 8.1).
- c) $[C, \eta]$ is in the Prym–Brill–Noether divisor in $\mathcal{R}_{6,2}$, i.e.

$$\emptyset \neq V_3(C, \eta) = \left\{ L \in \text{Nm}_f^{-1}(K_C) \mid h^0(\tilde{C}, L) \geq r+1, h^0(\tilde{C}, L) \equiv r+1 \pmod{2} \right\}$$

where $f: \tilde{C} \rightarrow C$ is the étale double cover associated to η ([FGSMV14], Theorem 0.4).

- d) $[C, \eta]$ is a section of a Nikulin surface (see [FV12], Theorem 0.5).

Remark 4.5. We can use the characterization of Lemma 4.3 to give another illustrative demonstration of the divisoriality of $\mathcal{B}_{6,\ell}$. This is achieved by explicitly constructing a pair $[C, \eta]$ and a line bundle $L \in W_6^2(C)$ such that the map (4.2) is an isomorphism. For brevity we skip the necessary proof that various moduli spaces of linear series and torsion points are irreducible.

The construction of $[C, \eta, L]$ is as follows. Let C be a plane quintic and choose any $L \in W_6^2(C)$. Let $\vartheta = \mathcal{O}_C(1)$ be the unique g_5^2 on C and recall that it is an odd theta characteristic. Now L can be written as $L = \vartheta \otimes \mathcal{O}_C(x)$ for some point $x \in C$. In particular, x is a base point of L . Now choose an ℓ -torsion bundle η on C such that $h^0(C, \vartheta \otimes \eta) = 0$. Then, by Riemann–Roch and Serre duality, $h^0(C, \vartheta \otimes \eta^{-1}) = 0$ as well. This implies

$$h^0(C, L \otimes \eta) = h^0(C, L \otimes \eta^{-1}) = 1$$

and x is neither a base point of $L \otimes \eta$ nor of $L \otimes \eta^{-1}$. Let $H^0(C, L \otimes \eta) = \langle \sigma \rangle$ and $H^0(C, L \otimes \eta^{-1}) = \langle \tau \rangle$ and consider the map

$$\langle \sigma \rangle \otimes \langle \tau \rangle \rightarrow H^0(C, L^{\otimes 2}) / \text{Sym}^2 H^0(C, L)$$

Observe that the multiplication map $\text{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$ is injective, since

$$\text{Sym}^2 H^0(C, \mathcal{O}_C(1)) \rightarrow H^0(C, \mathcal{O}_C(2))$$

is an isomorphism ($C \subseteq \mathbb{P}^2$ is not contained in a quadric). The base locus of the image of $\text{Sym}^2 H^0(C, L)$ in $H^0(C, L^{\otimes 2}) = H^0(C, \omega_C(2x))$ contains $2x$. But $\sigma \otimes \tau$, considered as a section in $H^0(C, L^{\otimes 2})$, does not vanish at x . Therefore it cannot be contained in the image of $\text{Sym}^2(C, L)$, whence

$$H^0(C, L^{\otimes 2}) \cong \langle \sigma \otimes \tau \rangle \oplus \text{Sym}^2 H^0(C, L)$$

and we are done.

5 Divisor classes

5.1 Strategy

An effective method to calculate divisor classes is to give a determinantal description of the divisors, i.e., express them as the locus where a certain morphism between vector bundles drops rank. If the divisor involves global sections of line bundles on curves, the vector bundles are usually constructed over some space \mathfrak{G}_d^r of linear series over the moduli space of curves.

To calculate the classes of $\mathcal{B}_{g,\ell}$ or some compactification of it, a direct approach would be to try to use Lemma 4.2 and globalize the map

$$\delta_{E_C \otimes \eta}: H^0(C, \omega_C \otimes A^{-1} \otimes \eta) \rightarrow H^1(C, A \otimes \eta)$$

to a morphism of vector bundles over $\mathfrak{G}_d^{r,(\ell)} = \mathfrak{G}_d^r \times_{\mathcal{M}_g} \mathcal{R}_{g,\ell}$. A naive attempt is to pass to the moduli stacks and to try to create a global extension

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{E} \rightarrow \omega_\chi \otimes \mathcal{A}^{-1} \rightarrow 0,$$

on the universal curve $\chi: \mathfrak{C}_d^{r,(\ell)} \rightarrow \mathfrak{G}_d^{r,(\ell)}$, tensor it by the universal ℓ -torsion bundle \mathcal{P} and use the map induced by the long exact sequence of the pushforward σ_* where $\sigma: \mathfrak{G}_d^{r,(\ell)} \rightarrow \mathcal{R}_{g,\ell}$.

However, this naive approach must fail. The bundle E_C , as an extension of $\omega_C \otimes A^{-1}$ by A , is only defined up to isomorphism on each curve and the choice can not be made globally on the whole moduli space. It is possible, though, to give a choice-free description of the condition that the boundary morphism induced by $E_C \otimes \eta$ is not an isomorphism.

To this end, let $L = \omega_C \otimes A^{-1}$ and observe that the codomain of $\delta_{E_C \otimes \eta}$ is

$$H^1(C, A \otimes \eta) \cong H^0(C, L \otimes \eta^{-1})^\vee$$

by Serre duality. Now the map

$$H^0(C, L \otimes \eta) \otimes \left(\frac{H^0(C, L^{\otimes 2})}{\text{Sym}^2 H^0(C, L)} \right)^\vee \rightarrow H^0(C, L \otimes \eta^{-1})^\vee \quad (5.1)$$

can be defined canonically by setting

$$s \otimes f \mapsto [t \mapsto f(s \cdot t)]$$

The quotient that appears in (5.1) can be seen as encoding the \mathbb{C}^* of possible choices for E_C in $\text{Ext}^1(L, A)$. It is clear that the map (5.1) is an isomorphism if and only if $\delta_{E_C \otimes \eta}$ is. Since there are no choices involved in defining the map, we can readily globalize it.

5.2 Definition of the degeneracy locus

We first construct an appropriate partial compactification of $\mathcal{R}_{g,\ell}$ where the class calculations can be carried out. Here we use a setup similar to [FL10] and [CEFS13]. Again denote by $\pi: \mathcal{R}_{g,\ell} \rightarrow \mathcal{M}_g$ the forgetful map. Let $\mathcal{R}'_{6,\ell} = \mathcal{R}_{6,\ell}^0 \cup \pi^*(\Delta_0^0)$, where $\mathcal{R}_{6,\ell}^0$ is the locus of smooth curves $[C, \eta]$ such that $\dim W_6^2(C) = 0$ and $H^1(C, L \otimes \eta) = 0$ for all $L \in W_6^2(C)$, and Δ_0^0 is the locus of irreducible one-nodal curves $[C_{p,q}] \in \Delta_0$ where $[C, p, q] \in \mathcal{M}_{5,2}$ is Petri general.

Similarly, let $\mathcal{R}'_{8,\ell} = \mathcal{R}_{8,\ell}^0 \cup \pi^*(\Delta_0^0)$ be the locus of smooth curves $[C, \eta] \in \mathcal{R}_{8,\ell}$ such that $\dim W_9^3(C) = 0$, and $H^1(C, L \otimes \eta) = 0$ for all $L \in W_9^3(C)$, while Δ_0^0 is the locus of curves $[C_{p,q}]$ with $[C, p, q] \in \mathcal{M}_{7,2}$ Petri general. Observe that in both cases the complement of $\mathcal{R}'_{g,\ell}$ in $\mathcal{R}_{g,\ell} \cup \pi^*(\Delta_0)$ has codimension 2, so divisor class calculations will not be affected (use Mumford's theorem, Theorem 4.1 in [Muk93] and the discussion in section 8 of [FGSMV14]).

We are now in a position to provide a determinantal description of the divisor $\mathcal{B}_{g,\ell}$. To this end, we will construct a morphism of vector bundles of the same rank over $\mathcal{R}'_{g,\ell}$ such that on fibers it corresponds exactly to the map in (5.1). Then $\overline{\mathcal{B}}_{g,\ell}$ will be contained in the first degeneracy locus of this morphism and its class can be calculated using Porteous' formula.

The setup is almost the same for both genera. In genus 6 we let $r = 2$, $d = 6$ and in genus 8 we let $r = 3$, $d = 9$. Now let $\mathfrak{G}_d^{r,(\ell)}$ be the moduli stack of triples $[C, \eta, L]$ over $\mathcal{R}'_{g,\ell}$ where $L \in W_d^r(C)$ and let $\sigma: \mathfrak{G}_d^{r,(\ell)} \rightarrow \mathcal{R}'_{g,\ell}$ be the morphism forgetting the g_d^r . Denote further by $\chi: \mathfrak{C}_d^{r,(\ell)} \rightarrow \mathfrak{G}_d^{r,(\ell)}$ the universal curve and let \mathcal{L} be the universal g_d^r . We also have the universal ℓ -torsion bundle \mathcal{P} over $\mathfrak{C}_d^{r,(\ell)}$. We will slightly abuse notation and denote the pullbacks of λ , δ'_0 , δ''_0 and $\delta_0^{(a)}$ by σ by the same symbols, respectively.

By Grauert's theorem, $\chi_*(\mathcal{L}^{\otimes i})$ is a vector bundle for $i = 1, 2$. However, we do not know this for $\chi_*(\mathcal{L} \otimes \mathcal{P})$, since the dimension of $H^0(C, L \otimes \eta)$ jumps on fibers over the whole boundary divisor Δ_0'' :

Lemma 5.1. *Let $g = 6$ and $[C, \eta] \in \Delta_0''$. Then for any $L \in W_6^2(C)$ we have $h^0(C, L \otimes \eta) = 2$. Likewise, for $g = 8$ and any $L \in W_9^3(C)$ on $[C, \eta] \in \Delta_0''$ we have $h^0(C, L \otimes \eta) = 3$.*

Proof. Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C and x be the node. Then $\nu^*\eta = \mathcal{O}_{\tilde{C}}$ and $\nu^*L \in W_d^r(\tilde{C})$, since \tilde{C} is Brill–Noether general. From the exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \nu_*\mathcal{O}_{\tilde{C}} \xrightarrow{\epsilon} \mathbb{C}_x \rightarrow 0$$

we get

$$0 \rightarrow L \otimes \eta \rightarrow v_* v^* L \xrightarrow{e'} L \otimes \eta|_x \rightarrow 0$$

and taking long exact sequence in cohomology we obtain

$$0 \rightarrow H^0(C, L \otimes \eta) \rightarrow H^0(\tilde{C}, v^* L) \xrightarrow{H^0(e')} \mathbb{C}$$

Now $H^0(e)$ is the zero map, hence $H^0(e')$ must be nonzero and we get

$$h^0(C, L \otimes \eta) = h^0(\tilde{C}, v^* L) - 1 = r + 1 - 1 = r \quad \blacksquare$$

This shows that $R^1 \chi_*(\mathcal{L} \otimes \mathcal{P})$ is supported on Δ'_0 and of rank 1 over there. On the other hand, we do not know whether $\chi_*(\mathcal{L} \otimes \mathcal{P})$ is a vector bundle. However, it is torsion-free, hence locally free in codimension 1 and we can throw out the at most codimension 2 loci in $\mathcal{R}'_{g,\ell}$ where the rank jumps. This will not affect our divisor class calculations. Hence we will assume $\chi_*(\mathcal{L} \otimes \mathcal{P})$ and $\chi_*(\mathcal{L} \otimes \mathcal{P}^{-1})$ are vector bundles. Now let

$$\mathcal{E} = \chi_*(\mathcal{L} \otimes \mathcal{P}) \otimes (\chi_*(\mathcal{L}^{\otimes 2}) / \text{Sym}^2 \chi_*(\mathcal{L}))^\vee$$

and

$$\mathcal{F} = (\chi_*(\mathcal{L} \otimes \mathcal{P}^{-1}))^\vee$$

We obtain a morphism

$$\phi_{g,\ell}: \mathcal{E} \rightarrow \mathcal{F} \quad (5.2)$$

whose first degeneracy locus $Z_1(\phi_{g,\ell})$, pushed forward by σ and restricted to the Mukai locus $\mathcal{R}_{g,\ell}^\mu$, coincides with our divisor $\mathcal{B}_{g,\ell}$.

5.3 Calculation of the classes

First we apply Porteous' formula to the morphism (5.2) to obtain

$$[Z_1(\phi_{g,\ell})] = c_1(\mathcal{F} - \mathcal{E}) = c_1(\mathcal{F}) - c_1(\mathcal{E})$$

Using the elementary fact

$$c_1(\text{Sym}^2 \mathcal{G}) = (\text{rk}(\mathcal{G}) + 1)c_1(\mathcal{G})$$

for a vector bundle \mathcal{G} we can write

$$c_1(\mathcal{E}) = c_1(\chi_*(\mathcal{L} \otimes \mathcal{P})) - (r-1)c_1(\chi_*(\mathcal{L}^{\otimes 2})) + (r-1)(r+2)c_1(\chi_*(\mathcal{L}))$$

We use Grothendieck–Riemann–Roch to calculate the Chern classes in these expressions. Let ω_χ be the relative dualizing sheaf of χ and consider the classes

$$\mathfrak{a} = \chi_*(c_1^2(\mathcal{L})), \quad \mathfrak{b} = \chi_*(c_1(\mathcal{L}) \cdot c_1(\omega_\chi)), \quad \mathfrak{c} = c_1(\chi_*(\mathcal{L}))$$

in $A^1(\mathfrak{G}_d^{r,(\ell)})$. Furthermore, let $\mathfrak{d} = c_1(\mathbb{R}^1\chi_*(\mathcal{L} \otimes \mathcal{P}))$. For brevity, set

$$\rho = \sum_{a=1}^{\lfloor \ell/2 \rfloor} \frac{a(\ell-a)}{\ell} \delta_0^{(a)}$$

Applying Grothendieck–Riemann–Roch and using [CEFS13], Proposition 1.6, we get

$$\begin{aligned} c_1(\chi_*(\mathcal{L} \otimes \mathcal{P}^{\pm 1})) &= \lambda + \frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\rho + \mathfrak{d} \\ c_1(\chi_*(\mathcal{L}^{\otimes 2})) &= \lambda + 2a - b \end{aligned}$$

Putting everything together, we obtain

$$[Z_1(\phi_{g,\ell})] = (r-3)\lambda + (2r-3)a - (r-2)b - (r^2+r-2)c - 2\mathfrak{d} + \rho \quad (5.3)$$

Lemma 5.2. *For $g = 6$ we have*

$$\sigma_*(a) = -93 \cdot \lambda + \frac{23}{2}\pi^*(\delta_0), \quad \sigma_*(b) = -\frac{3}{2}\lambda + \frac{3}{4}\pi^*(\delta_0), \quad \sigma_*(c) = -\frac{133}{4}\lambda + \frac{33}{8}\pi^*(\delta_0)$$

and for $g = 8$ we have

$$\sigma_*(a) = -267 \cdot \lambda + \frac{69}{2}\pi^*(\delta_0), \quad \sigma_*(b) = 3 \cdot \lambda + \frac{3}{2}\pi^*(\delta_0), \quad \sigma_*(c) = -100 \cdot \lambda + 13\pi^*(\delta_0)$$

Proof. Use the machinery of [Far09], in particular Lemma 2.6, Lemma 2.13 and Proposition 2.12. \blacksquare

Remark 5.3. A different choice of Poincaré bundle \mathcal{L} affects the classes a , b and c . However, the class of the degeneracy locus of $\phi_{g,\ell}$ is independent of this choice (see the discussion before Theorem 2.1 in [Far09]).

Now we only need to pushforward $[Z_1(\phi_{g,\ell})]$ by σ to $\mathcal{R}'_{g,\ell}$, which has the effect of multiplying the coefficients of λ , δ_0'' and $\delta_0^{(a)}$ in (5.3) by the degree of σ . This degree is 5 in the case of $g = 6$ and 14 in the case of $g = 8$ (the respective number of \mathfrak{g}_d^r on the general curve). Plug in the expressions of Lemma 5.2 to obtain:

Theorem 5.4. *The class of the degeneracy locus $\sigma_*Z_1(\phi_{6,\ell})$ is*

$$[\overline{\mathcal{B}}_{6,\ell}]^{\text{virt}} = 35\lambda - 5(\delta_0' + 3\delta_0'') - \frac{5}{\ell} \sum_{a=1}^{\lfloor \ell/2 \rfloor} (\ell^2 - a\ell + a^2)\delta_0^{(a)}$$

Theorem 5.5. *The class of the degeneracy locus $\sigma_*Z_1(\phi_{8,\ell})$ is*

$$[\overline{\mathcal{B}}_{8,\ell}]^{\text{virt}} = 196\lambda - 28(\delta_0' + 2\delta_0'') - \frac{14}{\ell} \sum_{a=1}^{\lfloor \ell/2 \rfloor} (2\ell^2 - a\ell + a^2)\delta_0^{(a)}$$

In particular, since $\sigma_* Z_1(\phi_{g,\ell}) \cap \mathcal{R}_{g,\ell} = \mathcal{B}_{g,\ell}$, the class $[\overline{\mathcal{B}}_{g,\ell}]^{\text{virt}} - n[\overline{\mathcal{B}}_{g,\ell}]$ is effective and entirely supported on the boundary of $\mathcal{R}'_{g,\ell}$ for some $n \geq 1$.

Remark 5.6. The morphism $\phi_{g,\ell}$ is degenerate over the boundary component Δ_0'' , with order 1 for $g = 6$ and order 2 for $g = 8$. We can therefore subtract an additional $5\delta_0''$ and $28\delta_0''$, respectively.

Remark 5.7. The coefficients appearing in the expression of $\overline{\mathcal{B}}_{6,\ell}$ are divisible by 5, which is exactly the degree of the map $\sigma: \mathfrak{G}_6^{2,(\ell)} \rightarrow \mathcal{R}'_{6,\ell}$. This can be explained by observing that the boundary morphism (4.1) fails to be an isomorphism for some $A \in W_4^1(C)$ if and only if $H^0(C, E_C \otimes \eta) \neq 0$. But since E_C does not depend on the choice of A , the morphism surprisingly fails to be bijective for *all* $A \in W_4^1(C)$.

Similarly, the coefficients for $\overline{\mathcal{B}}_{8,\ell}$ are divisible by $28 = 2 \cdot 14$, where $14 = \deg(\sigma)$. Observe that by Serre duality, $\chi(E_C \otimes \eta) = 0$, and by the isomorphism $E_C^\vee \otimes \omega_C \cong E_C$, we have $H^0(C, E_C \otimes \eta) = 0$ if and only if $H^0(C, E_C \otimes \eta^{-1}) = 0$. This explains the additional factor of two.

6 Application to the birational geometry of modular varieties

6.1 An improvement of existing divisor classes

Recall the following result:

Theorem 6.1 ([CEFS13], Theorem 0.7). *Set $g = 2i + 2 \geq 4$ and $\ell \geq 3$ such that $i \equiv 1 \pmod{2}$ or $\binom{2i-1}{i} \equiv 0 \pmod{2}$. The virtual class of the closure in $\mathcal{R}'_{g,\ell}$ of the locus $\mathcal{D}_{g,\ell}$ of level ℓ curves $[C, \eta] \in \mathcal{R}_{g,\ell}$ such that $K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) \neq 0$ is equal to*

$$[\overline{\mathcal{D}}_{g,\ell}]^{\text{virt}} = \frac{1}{i-1} \binom{2i-2}{i} \left((6i+1)\lambda - i(\delta'_0 + \delta''_0) - \frac{1}{\ell} \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} (i\ell^2 + 5a^2i - 5a\ell - 2a^2 + 2a\ell)\delta_0^{(a)} \right)$$

Here $K_{i,j}(C, \eta^{\otimes(\ell-2)}, K_C \otimes \eta)$ is the Koszul cohomology group defined in [Gre84]. We quickly sketch how this result was obtained. For a globally generated vector bundle E on C , let M_E be the kernel of the surjective evaluation map

$$H^0(C, E) \otimes \mathcal{O}_C \rightarrow E \rightarrow 0$$

Then by standard arguments, e.g. [AN10], we have an identification

$$K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) = H^0(C, \wedge^i M_{K_C \otimes \eta} \otimes K_C \otimes \eta^{-1})$$

This in turn can be identified with the kernel of the map

$$\wedge^i H^0(C, K_C \otimes \eta) \otimes H^0(C, K_C \otimes \eta^{-1}) \rightarrow H^0(C, \wedge^{i-1} M_{K_C \otimes \eta} \otimes K_C^{\otimes 2})$$

Note that the domain and the target are vector spaces of the same dimension. This map is then globalized to a map χ between vector bundles of the same rank over $\mathcal{R}'_{g,\ell}$ and its first degeneracy locus can be calculated using Porteous' formula to obtain the class of Theorem 6.1.

We will now show that χ is degenerate along the boundary divisors $\Delta_0^{(a)}$ by calculating a lower bound on the dimension of $V := H^0(X, \wedge^i M_{\omega_X \otimes \eta} \otimes \omega_X \otimes \eta^{-1})$ for a general curve $[X, \eta] \in \Delta_0^{(a)}$. The result does not depend on a .

Let $X = C \cup_{p,q} E$ where $E \cong \mathbb{P}^1$ is exceptional. Observe that $\omega_X|_E = \mathcal{O}_E$ while $\omega_X|_C = K_C(p+q)$. One then calculates that

$$M_{\omega_X \otimes \eta}|_C = M_{K_C(p+q) \otimes \eta_C} \text{ and } M_{\omega_X \otimes \eta}|_E = \mathcal{O}_E(-1) \oplus \mathcal{O}_E^{\oplus(g-3)}$$

We let $M := M_{K_C(p+q) \otimes \eta_C}$. By the Mayer–Vietoris sequence V is the kernel of

$$\begin{aligned} H^0(C, \wedge^i M \otimes K_C(p+q) \otimes \eta_C^{-1}) \oplus H^0(E, \wedge^i (\mathcal{O}_E(-1) \oplus \mathcal{O}_E^{\oplus(g-3)}) \otimes \mathcal{O}_E(-1)) \\ \rightarrow H^0(\wedge^i M_{\omega_X \otimes \eta} \otimes \omega_X \otimes \eta^{-1}|_{p+q}) \end{aligned}$$

Since $\wedge^i M_{\omega_X \otimes \eta}$ has rank $\binom{2i}{i}$, the latter space has dimension $2 \cdot \binom{2i}{i}$, while the bundle on E has no sections. Using Riemann–Roch, we calculate

$$\begin{aligned} h^0(C, \wedge^i M \otimes K_C(p+q) \otimes \eta_C^{-1}) &\geq -(2g-3) \binom{2i-1}{i-1} + \binom{2i}{i} (2g-1) + \binom{2i}{i} (2-g) \\ &= 5 \binom{2i-1}{i-1} \end{aligned}$$

hence the kernel has dimension at least

$$5 \binom{2i-1}{i-1} - 2 \binom{2i}{i} = \binom{2i-1}{i-1}$$

and therefore χ is degenerate to this order on the boundary Δ_0^{ram} . We have proved:

Proposition 6.2. *The divisor class $[\overline{\mathcal{D}}_{g,\ell}]^{\text{virt}} - \binom{2i-1}{i-1} \sum_{a=1}^{\lfloor \ell/2 \rfloor} \delta_0^{(a)}$ is effective.*

Example 6.3. For $g = 8$ (i.e. $i = 3$) and $\ell = 3$ we obtain that

$$[\overline{\mathcal{D}}_{8,3}]^{\text{virt}} - 10\delta_0^{(1)} \equiv 38\lambda - 6(\delta'_0 + \delta''_0) - \frac{32}{3}\delta_0^{(1)}$$

is effective.

6.2 Degeneracy of $\overline{\mathcal{B}}_{8,3}$ on the boundary

Recall that our strategy to calculate the divisor class of $\overline{\mathcal{B}}_{8,3}$ was to globalize the map

$$H^0(C, L \otimes \eta) \otimes \left(\frac{H^0(C, L^{\otimes 2})}{\text{Sym}^2 H^0(C, L)} \right)^\vee \rightarrow H^0(C, L \otimes \eta^{-1})^\vee$$

where $L \in W_9^3(C)$. Using Tensor-Hom adjunction, this map fails to be injective if and only if the bilinear map corresponding to the multiplication map

$$\mu_{[C, \eta, L]}: H^0(C, L \otimes \eta) \otimes H^0(C, L \otimes \eta^{-1}) \rightarrow H^0(C, L^{\otimes 2}) / \text{Sym}^2 H^0(C, L) \quad (6.1)$$

is degenerate. We will in fact show that for general $[X, \eta] \in \Delta_0^{(1)}$ and $L \in W_9^3(X)$ the map $\mu_{[X, \eta, L]}$ is the zero map, i.e., the image of $H^0(X, L \otimes \eta) \otimes H^0(X, L \otimes \eta^{-1}) \rightarrow H^0(X, L^{\otimes 2})$ is contained in the image of $\text{Sym}^2 H^0(X, L)$.

A line bundle $L \in W_9^3(X)$ can be described as follows. The restriction $L_C = L|_C$ of L to C has the property $h^0(C, L_C(-p-q)) = 3$. Since L restricts to \mathcal{O}_E on E , any global section $s \in H^0(X, L)$ restricts to a constant on E and hence the restriction to C has the same value at p and q . If $s|_E = 0$, then $s|_C \in H^0(C, L_C(-p-q))$. If $s|_E$ is instead a nonzero constant then $s|_C$ is a global section of L_C which vanishes neither at p nor at q . Fix such a section σ . Then we have an isomorphism

$$H^0(X, L) \cong H^0(C, L_C(-p-q)) \oplus \langle \sigma \rangle$$

We get

$$\text{Sym}^2 H^0(X, L) \cong \text{Sym}^2 H^0(C, L_C(-p-q)) \oplus \langle \sigma \otimes \sigma \rangle \oplus (\langle \sigma \rangle \otimes H^0(C, L_C(-p-q))) \quad (6.2)$$

and for $[C, p, q]$ general the map

$$\text{Sym}^2 H^0(X, L) \rightarrow H^0(X, L^{\otimes 2})$$

is injective with one-dimensional cokernel. On the other hand, by Riemann–Roch, we have $\dim H^0(X, L^{\otimes 2}(-p-q)) = 9$ for the space of sections vanishing at p and q . Comparing this with the expression (6.2) we see that all these sections come from $\text{Sym}^2 H^0(X, L)$.

Now we consider the space $H^0(X, L \otimes \eta^{-1})$. On E the line bundle $L \otimes \eta^{-1}$ restricts to $\mathcal{O}_E(-1)$ and on C to $L_C \otimes \eta_C^{-1}$. Since $H^0(E, \mathcal{O}_E(-1)) = 0$ we have the identity

$$H^0(X, L \otimes \eta^{-1}) = H^0(C, L_C \otimes \eta_C^{-1} \otimes \mathcal{O}_C(-p-q))$$

so all sections here vanish at p and q . This implies that the multiplication map

$$H^0(X, L \otimes \eta) \otimes H^0(X, L \otimes \eta^{-1}) \rightarrow H^0(X, L^{\otimes 2})$$

factors through $H^0(X, L^{\otimes 2}(-p-q))$ and hence through the image of $\text{Sym}^2 H^0(X, L)$. This means that the multiplication map $\mu_{[X, \eta, L]}$ is indeed zero. We have proved:

Proposition 6.4. *The morphism $\phi: \mathcal{E} \rightarrow \mathcal{F}$ of (5.2) between vector bundles on $\mathcal{G}_9^{3,(3)}$ is degenerate to order 2 over $\Delta_0^{(1)}$. Hence $[Z_1(\phi)] - 2\delta_0^{(1)}$ is effective and therefore*

$$[\overline{\mathcal{B}}_{8,3}]^{\text{virt}} - 28\delta_0^{(1)} = 196\lambda - 28(\delta'_0 + 2\delta''_0) - \frac{308}{3}\delta_0^{(1)}$$

is effective as well.

Theorem 6.5. $\overline{\mathcal{R}}_{8,3}$ is of general type.

Proof. We take the effective linear combination

$$\begin{aligned} \frac{1}{119}([\overline{\mathcal{B}}_{8,3}]^{\text{virt}} - 28\delta_0^{(1)}) + \frac{5}{17}([\overline{\mathcal{D}}_{8,3}] - 10\delta_0^{(1)}) &\leq \frac{218}{17}\lambda - 2(\delta'_0 + \delta''_0) - 4\delta_0^{(1)} \\ &= K_{\mathcal{R}'_{8,3}} - \frac{3}{17}\lambda \end{aligned}$$

hence $K_{\mathcal{R}'_{8,3}}$ is big. Now we invoke Remark 3.5 from [CEFS13] to show that the same holds for $K_{\overline{\mathcal{R}}_{8,3}}$. ■

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