

The normal bundle of canonical genus 8 curves

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Abstract

We study the stability of the normal bundle of canonical genus 8 curves and prove that on a general curve the bundle is stable. The proof rests on Mukai's description of these curves as linear sections of a Grassmannian $G(2, 6)$. This is the next case of a conjecture by M. Aprodu, G. Farkas, and A. Ortega: the general canonical curve of every genus $g \geq 7$ should have stable normal bundle. We also give some more evidence for this conjecture in higher genus.

1 Introduction

To an embedded curve $C \hookrightarrow \mathbb{P}^r$ with $L = \mathcal{O}_C(1)$ one can naturally attach two distinguished vector bundles. First there is the restricted cotangent bundle $\Omega_{\mathbb{P}^r}(1)|_C$ which by the Euler sequence coincides with the kernel bundle M_L in the evaluation exact sequence

$$0 \rightarrow M_L \rightarrow H^0(C, L) \otimes \mathcal{O}_C \rightarrow L \rightarrow 0.$$

A lot is known about the geometry of M_L . For instance, it governs the syzygies of the embedding of C , and a simple argument (like in [ES12]) shows that in most cases M_L is a stable vector bundle. In particular this is true for the canonical embedding, i.e., for $L = \omega_C$.

The second distinguished vector bundle is the normal bundle $\mathcal{N}_{C/\mathbb{P}^r}$ of the embedding. Of particular interest is again the normal bundle $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ of the canonical embedding of C . As it turns out, the study of normal bundles is much more delicate than the study of the restricted cotangent bundles. In particular, not much is known about stability properties, not even for canonical curves.

Of course a non-hyperelliptic canonical curve of genus 3 is a plane quartic, hence the normal bundle is a line bundle and therefore stable. Moving on, canonical curves of genera 4 and 5 are complete intersections, hence they do not have stable normal bundles. In [AFO16] it was shown that a tetragonal curve of genus at least 6 has unstable normal bundle. This applies in particular to all canonical curves of genus 6.

On the other hand, in the same paper [AFO16] the authors prove the first nontrivial positive result: in genus 7 the general canonical curve has stable normal bundle. Observe

that, starting from genus 7, the general curve is no longer tetragonal. Based on these considerations, the authors then made the following conjecture:

Conjecture 1.1 ([AFO16, Conjecture 0.4]). *The normal bundle of a general canonical curve of genus $g \geq 7$ is stable.*

In this paper we consider the next case, genus 8, and provide a full solution:

Theorem 1.2. *The normal bundle of a canonical curve C of genus 8 is stable if and only if the curve does not have a g_7^2 . Furthermore, it is polystable if and only if the curve is not tetragonal.*

Note that a tetragonal curve always has a g_7^2 [Muk93, Lemma 3.8].

We sketch the proof of Theorem 1.2. The first ingredient is a construction of S. Mukai, found in [Muk93], showing that the general genus 8 curve is a transversal linear section of a Grassmannian $G = G(2, 6)$ in its Plücker embedding. The normal bundle of C then arises as the pullback of the normal bundle of G , which has a description in terms of the tautological bundle S on G . More precisely, we have $\mathcal{N}_{C/\mathbb{P}^7}(-1) = \bigwedge^2 S_C^\vee$, where S_C is the restriction of S to C .

In Section 2.4 we then prove that S_C is stable. Here we again use essential properties of the Mukai embedding $C \subseteq G$ to describe the geometry of S_C . Since exterior powers of stable bundles are polystable, we are left with proving that $\mathcal{N}_{C/\mathbb{P}^7}$ does not decompose. We achieve this in Section 2.5 through a case-by-case analysis.

If $W_7^2(C) \neq \emptyset$ but the curve is not tetragonal, we show in Section 3 that the normal bundle is not stable, but still polystable. The essential ingredient is the existence of complete intersection models of these curves in $\mathbb{P}^2 \times \mathbb{P}^2$, constructed by M. Ide and S. Mukai in [IM03]. Interestingly, for a suitable limit definition of S_C , the description of $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ as $\bigwedge^2 S_C^\vee$ still holds.

Finally, in two appendices, we first reprove the result that no canonical genus 6 curve has stable normal bundle using Mukai's Grassmannian construction. Secondly, we give some remarks about the case which comes next, i.e., genus 9 canonical curves, which seems to be considerably more difficult and might require new techniques.

Remark 1.3. For more general linear systems than the canonical one, not much is known either. On the one hand, embeddings of low degree, as well as certain embeddings by non-complete linear series, are expected to be unstable. Normal bundles of rational and elliptic space curves have been studied and shown to be unstable in many cases, for instance in [EV82; HS83].

On the other hand, at least conjecturally, embeddings by complete linear systems of high degree are expected to be stable. In [EL92], Ein and Lazarsfeld prove that normal bundles of elliptic curves in an embedding of degree at least 3 are polystable and they show that for genus $g \geq 2$ and degree $d \gg 0$ the normal bundles are at least not too unstable.

Remark 1.4. The question about stability properties of normal bundles can also be posed for polarized K3 surfaces. There is an intimate relation between the geometry of K3s and the geometry of canonical curves. In particular, Mukai's constructions originally were set

up for Fano 3-folds and K3 surfaces, with curves as an application (see [Muk88]). Using this, we can almost immediately lift results about normal bundles up to genus 9 from a canonical curve C to a polarized K3 surface $(S, \mathcal{O}_S(C))$. It may be worth investigating the opposite direction: consider stability of normal bundles of polarized K3 surfaces and try to infer something about the normal bundle of their hyperplane sections.

Acknowledgements The content of this article was part of my Ph.D. thesis. First I would like to thank my Ph.D. advisor Gavril Farkas for suggesting this very interesting problem to me. Thanks also goes to Carel Faber, my co-advisor, who hosted me at Universiteit Utrecht during the time this paper was conceived. I greatly enjoyed the atmosphere and guidance there. The research stay was generously supported by the IRTG 1800 of the Deutsche Forschungsgemeinschaft.

2 Stability of the normal bundle for general curves

2.1 Facts about normal bundles of canonical curves

Consider a non-hyperelliptic smooth curve of genus $g \geq 3$, embedded by the canonical embedding in \mathbb{P}^{g-1} . We denote the normal bundle of this embedding by $\mathcal{N}_{C/\mathbb{P}^{g-1}}$. Using the Euler exact sequence and the normal bundle exact sequence we calculate that in this case

$$\det(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = \omega_C^{\otimes(g+1)}, \quad \deg(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = (g+1)(2g-2)$$

and we remark that $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ is of rank $g-2$.

To discuss stability properties of $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ in a more precise way, recall the following definition:

Definition 2.1. A vector bundle on a curve C is called *polystable* if it splits into the direct sum of stable bundles, all of the same slope.

Of course every stable bundle is polystable and polystable bundles are semistable.

We now consider, as an easy example, the stability of the normal bundle of canonical curves in low genus, briefly mentioned already in the introduction.

Every non-hyperelliptic genus 3 curve is canonically embedded as a quartic in \mathbb{P}^2 , hence the normal bundle is the line bundle $\mathcal{O}_C(4)$ and therefore stable. For genus 4 and 5, the general canonical curve is the complete intersection of a cubic and a quadric, or three quadrics, respectively. Since the normal bundle of a complete intersection splits, we get

$$\mathcal{N}_{C/\mathbb{P}^{g-1}} = \begin{cases} \mathcal{O}_C(2) \oplus \mathcal{O}_C(3), & g = 4 \\ \mathcal{O}_C(2) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2), & g = 5. \end{cases}$$

We see that for genus 4 the normal bundle is unstable while it is polystable (but not stable) for genus 5.

2.2 Kernel bundles

Now we introduce the main technical tool we need. If L is a line bundle on C then we denote by M_L the kernel of the evaluation morphism

$$H^0(C, L) \otimes \mathcal{O}_C \rightarrow L. \quad (2.1)$$

As a subsheaf of the (free) vector bundle $H^0(C, L) \otimes \mathcal{O}_C$, the sheaf M_L is itself a vector bundle and is called the *kernel bundle* of L . We denote its dual by $Q_L = M_L^\vee$. If L is globally generated, then the evaluation map (2.1) is surjective and the sequence

$$0 \rightarrow M_L \rightarrow H^0(C, L) \otimes \mathcal{O}_C \rightarrow L \rightarrow 0$$

is also exact on the right. We obtain the dual short exact sequence

$$0 \rightarrow L^{-1} \rightarrow H^0(C, L)^\vee \otimes \mathcal{O}_C \rightarrow Q_L \rightarrow 0$$

which shows that Q_L is globally generated. Taking global sections in this exact sequence also shows $h^0(C, Q_L) \geq h^0(C, L)$ and we expect equality to hold. We furthermore have that $\text{rk}(Q_L) = h^0(C, L) - 1$ and $\det Q_L = L$.

The bundles Q_L are often stable and in many cases we have a natural isomorphism $H^0(C, L)^\vee \cong H^0(C, Q_L)$. In this way, kernel bundles provide many examples of stable bundles with unusually many global sections. They have been used to construct vector bundles with special properties on curves and surfaces, e.g. Ulrich bundles ([ES03]), and bundles violating Mercat's conjecture ([FO12]).

Kernel bundles have also been used extensively in the context of Koszul cohomology, for instance in construction of Koszul divisors on moduli spaces of curves (see [Far09]). R. Lazarsfeld used them to show that a general curve on a general K3 surface satisfies the Brill–Noether–Petri theorem [Laz86]. C. Voisin made use of Kernel bundles in [Voi02; Voi05] to prove Green's conjecture for generic curves of even and odd genus, respectively.

2.3 Description of the normal bundle

Throughout the rest of this section we will assume that C is a curve of genus $g = 8$, canonically embedded in \mathbb{P}^7 , such that $W_7^2(C) = \emptyset$. In other words, C should be Brill–Noether general. By the work of Mukai ([Muk93]), such a curve C is a transversal linear section of a Grassmannian $G = G(2, 6)$, embedded by the Plücker embedding in \mathbb{P}^{14} . More precisely, there is a 7-plane $\mathbb{P}^7 \subset \mathbb{P}^{14}$ such that $C = G \cap \mathbb{P}^7$.

The inclusion $C \hookrightarrow G$ is induced by the global sections of the *Mukai bundle* E_C on C , an up to isomorphism uniquely defined stable rank 2 bundle with $h^0(C, E_C) = 6$. If ζ is any \mathfrak{g}_5^1 on C and $\eta = \omega_C \otimes \zeta^{-1} \in W_9^3(C)$ its Serre dual, then E_C sits in the exact sequence

$$0 \rightarrow \zeta \rightarrow E_C \rightarrow \eta \rightarrow 0$$

which is split on global sections.

The normal bundle $\mathcal{N}_{C/\mathbb{P}^7}$ has rank 6 and determinant $\omega_C^{\otimes 9}$. We can use the Grassmannian embedding of C to derive more information about the structure of $\mathcal{N}_{C/\mathbb{P}^7}$. Because $C = \mathbb{P}^7 \cap G$, we have the split exact sequence

$$0 \rightarrow \mathcal{N}_{C/G} \rightarrow \mathcal{N}_{C/\mathbb{P}^{14}} \rightarrow \mathcal{N}_{C/\mathbb{P}^7} \rightarrow 0$$

and the exact sequence of normal bundles induced by the inclusions $C \hookrightarrow G \hookrightarrow \mathbb{P}^{14}$

$$0 \rightarrow \mathcal{N}_{C/G} \rightarrow \mathcal{N}_{C/\mathbb{P}^{14}} \rightarrow \mathcal{N}_{G/\mathbb{P}^{14}}|_C \rightarrow 0$$

which together imply $\mathcal{N}_{C/\mathbb{P}^7} = \mathcal{N}_{G/\mathbb{P}^{14}}|_C$. But the normal bundle of a $G(2, n)$ in its Plücker embedding is explicitly given by

$$\mathcal{N}_{G(2,n)/\mathbb{P}^N} = \bigwedge^2 S^\vee \otimes \bigwedge^2 Q = \bigwedge^2 S^\vee \otimes \mathcal{O}_{G(2,n)}(1).$$

A proof of this classical fact can be found, e.g., in [Man98]. Here S is the tautological bundle over $G(2, n)$ and Q the universal quotient bundle of rank 2, sitting in the exact sequence $0 \rightarrow S \rightarrow \mathcal{O}_{G(2,n)}^{\oplus n} \rightarrow Q \rightarrow 0$. Hence if S_C denotes the restriction of S to C then

$$\mathcal{N}_{C/\mathbb{P}^7}(-1) = \bigwedge^2 S_C^\vee.$$

In what follows we will prove stability of this twist $\mathcal{N}_{C/\mathbb{P}^7}(-1)$. Note that $\det \mathcal{N}_{C/\mathbb{P}^7}(-1) = \omega_C^{\otimes 3}$ and the slope of this bundle is $\mu(\mathcal{N}_{C/\mathbb{P}^7}(-1)) = 7$.

Remark 2.2. Several things are special about the case of genus 8. In fact, this is the only genus $g \geq 6$ where the slope of the normal bundle is an integer. This happens if and only if

$$(g-2) \mid 2(g-1)(g+1)$$

and since $(g-2)$ and $(g-1)$ have no common factors, we must have $(g-2) \mid 2(g+1)$ which implies that all divisors of $(g-2)$ are 1, 2 or 3. So the only possibilities are $g = 3, 4, 5, 8$.

By the above explicit description of $\mathcal{N}_{C/\mathbb{P}^7}$ we also see that the normal bundle of a general canonical genus 8 curve is self-dual up to twist, since for any vector bundle \mathcal{F} of rank 4 we have the duality

$$\bigwedge^2 \mathcal{F} = \left(\bigwedge^2 \mathcal{F} \right)^\vee \otimes \det \mathcal{F}$$

and hence

$$\mathcal{N}_{C/\mathbb{P}^7}^\vee(2) \cong \mathcal{N}_{C/\mathbb{P}^7}(-1).$$

By comparing the degrees of $\mathcal{N}_{C/\mathbb{P}^{g-1}}(k)$ and $\mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(l)$ for $k, l \in \mathbb{Z}$ and arbitrary genus, we see that $g = 8$ is the only case where this could happen.

Our strategy is then to derive the stability of $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ from the stability of S_C . Hence we first have to understand the bundle S_C better. First of all noting some basic numerical

facts, the slope is $\mu(S_C^\vee) = 7/2$. We can restrict the universal exact sequence over the Grassmannian $G(2,6)$ to C and obtain

$$0 \rightarrow S_C \rightarrow \mathcal{O}_C^{\oplus 6} \rightarrow E_C \rightarrow 0$$

where E_C denotes, as before, the Mukai bundle of C . This implies that S_C^\vee is globally generated and $h^0(C, S_C^\vee) \geq 6$. We also have $\det(S_C^\vee) = \omega_C$ and $\chi(S_C^\vee) = -14$, hence $h^0(S_C^\vee) = h^1(S_C^\vee) - 14$.

2.4 Stability of the tautological bundle

We are going to prove that the restriction of the tautological bundle S of $G(2,6)$ to C is stable. The following lemma will be instrumental in what follows.

Lemma 2.3. *Let $\zeta \in W_5^1(C)$. Then there is a unique surjection $S_C^\vee \rightarrow \zeta$. This gives rise to an exact sequence*

$$0 \rightarrow Q_\eta \rightarrow S_C^\vee \rightarrow \zeta \rightarrow 0$$

where Q_η is the dual of the kernel bundle of $\eta = \omega_C \otimes \zeta^{-1}$.

Proof. The space $H^0(C, E_C \otimes \zeta^{-1}) = H^1(C, E_C \otimes \zeta)^\vee$ is easily seen to be 1-dimensional (for instance by Lemma 3.10 of [Muk93]), hence $h^0(C, E_C \otimes \zeta) = 11$. Let $V = H^0(C, E_C)$. Then by tensoring the exact sequence

$$0 \rightarrow S_C \rightarrow V \otimes \mathcal{O}_C \rightarrow E_C \rightarrow 0$$

with ζ and taking cohomology we obtain

$$0 \rightarrow H^0(C, S_C \otimes \zeta) \rightarrow V \otimes H^0(C, \zeta) \rightarrow H^0(C, E_C \otimes \zeta).$$

We want to show that the kernel $H^0(C, S_C \otimes \zeta) = \text{Hom}(S_C^\vee, \zeta)$ of the multiplication map $\mu: V \otimes H^0(C, \zeta) \rightarrow H^0(C, E_C \otimes \zeta)$ is one-dimensional. Using the base point free pencil trick we write

$$0 \rightarrow \zeta^{-1} \rightarrow H^0(C, \zeta) \otimes \mathcal{O}_C \rightarrow \zeta \rightarrow 0$$

and tensor this by E_C . Taking cohomology, we see that the kernel of μ is exactly $H^0(C, E_C \otimes \zeta^{-1})$, i.e., $\dim(\ker(\mu)) = 1$.

This means there is a unique nonzero morphism $S_C^\vee \rightarrow \zeta$ which surjects onto a line bundle of degree $d \leq \deg(\zeta) = 5$. This line bundle, as a quotient of S_C^\vee , must be globally generated and since $W_4^1(C) = \emptyset$ it must be ζ itself. We get an exact sequence

$$0 \rightarrow F \rightarrow S_C^\vee \rightarrow \zeta \rightarrow 0$$

where F is a rank 3 vector bundle with $\det(F) = \eta$, the Serre dual of ζ . Because $H^0(C, E_C) = H^0(C, \zeta) \oplus H^0(C, \eta)$ we get the following commutative diagram with exact columns and the upper two rows also exact:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \zeta^{-1} & \longrightarrow & H^0(C, \zeta) \otimes \mathcal{O}_C & \longrightarrow & \zeta \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & S_C & \longrightarrow & H^0(C, E) \otimes \mathcal{O}_C & \longrightarrow & E_C \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & F^\vee & \longrightarrow & H^0(C, \eta) \otimes \mathcal{O}_C & \longrightarrow & \eta \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

The Snake Lemma implies that the last row is exact as well, i.e., F^\vee is actually M_η . \blacksquare

Remark 2.4. This shows in particular that $h^0(C, S_C^\vee) = 6$, since from the exact sequence $0 \rightarrow Q_\eta \rightarrow S_C^\vee \rightarrow \zeta \rightarrow 0$ we get the bound

$$h^0(C, S_C^\vee) \leq h^0(C, \zeta) + h^0(C, Q_\eta) = 2 + 4$$

and we already knew $h^0(C, S_C^\vee) \geq 6$ from before.

The bundle Q_η that appears in Lemma 2.3 plays an important role in understanding the stability of S_C^\vee . It is not too hard to show that Q_η is itself stable.

Lemma 2.5. *Let $\eta \in W_9^3(C)$ and let Q_η be the dual of the kernel bundle of η . Then Q_η is stable and $H^0(C, Q_\eta) = H^0(C, \eta)^\vee$.*

Proof. The stability is proved in [ES12].

Let D be the intersection of C and a quadrisecant line in \mathbb{P}^3 where C is embedded by $|\eta|$. Then $\eta(-D) = \zeta \in W_5^1(C)$. We have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \eta^{-1} & \longrightarrow & H^0(\eta)^\vee \otimes \mathcal{O}_C & \longrightarrow & Q_\eta \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \zeta^{-1} & \longrightarrow & H^0(\zeta)^\vee \otimes \mathcal{O}_C & \longrightarrow & \zeta \longrightarrow 0
\end{array}$$

and an exact sequence

$$0 \rightarrow G \rightarrow Q_\eta \rightarrow \zeta \rightarrow 0$$

where G is a rank 2 bundle with $\det(G) = \mathcal{O}_C(D)$. Since $h^0(C, Q_\eta) \geq 4$ we have $h^0(C, G) \geq 2$.

Now G is the extension

$$0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0$$

of two line bundles A, B where we can assume $\deg(A) \geq 0$. By stability of Q_η we have $\deg(A) \leq 2$ and hence $2 \leq \deg(B) \leq 4$. In any case, $h^0(C, A) = h^0(C, B) = 1$ and hence $h^0(C, G) = 2$. This implies $h^0(C, Q_\eta) = 4$. ■

The following lemma is well-known.

Lemma 2.6. *Let G be a globally generated vector bundle on C with $H^0(C, G^\vee) = 0$. Then $h^0(C, \det(G)) \geq \operatorname{rk}(G) + 1$ with equality if and only if G is the dual of the kernel bundle of $\det(G)$.*

Proof. Since G can be generated by $\operatorname{rk}(G) + 1$ sections, we have a surjection $\mathcal{O}_C^{\operatorname{rk}(G)+1} \rightarrow G \rightarrow 0$ whose kernel is $\det(G)^{-1}$. Dualizing this sequence, we get

$$0 \rightarrow G^\vee \rightarrow \mathcal{O}_C^{\operatorname{rk}(G)+1} \rightarrow \det(G) \rightarrow 0$$

and using $H^0(C, G^\vee) = 0$ the result follows. ■

This result can be improved, see for instance Proposition 3.3 and Lemma 3.9 in [PR88]. We are now in a position to show that the tautological bundle is stable.

Proposition 2.7. *S_C^\vee is stable.*

Proof. Recall that S_C^\vee is globally generated of slope $\mu(S_C^\vee) = 7/2$. Let

$$0 \rightarrow F \rightarrow S_C^\vee \rightarrow M \rightarrow 0 \tag{2.2}$$

be an exact sequence of vector bundles. We distinguish several possibilities, depending on the rank of M . In each case we have to prove that $\mu(M) > \mu(S_C^\vee) = \frac{7}{2}$.

- $\operatorname{rk}(M) = 1$: Since S_C^\vee is globally generated, so is M . Hence $h^0(C, M) \geq 2$ which implies $\mu(M) = \deg(M) \geq 5$, for $W_4^1(C) = \emptyset$.
- $\operatorname{rk}(M) = 2$: Again M is generated by global sections and we have $H^0(C, M^\vee) = 0$ by dualizing the exact sequence (2.2). Using Lemma 2.6 we have $h^0(C, \det M) \geq 3$ which implies $\deg(M) \geq 8$ (there is no g_7^2 on C). Hence $\mu(M) \geq 4$.
- $\operatorname{rk}(M) = 3$: Now F is a line bundle, more concretely $F = \omega_C \otimes L^{-1}$ where $L = \det(M)$. Again by Lemma 2.6 we have $h^0(C, L) \geq 4$. If the inequality is strict, then $\deg(M) \geq 11$ since C has no g_{10}^4 . Hence $\mu(M) \geq \frac{11}{3} > \frac{7}{2}$ and we are done.

So assume $h^0(C, L) = 4$. Then M is actually the dual Q_L of the kernel bundle of L . Since $h^0(C, F) + h^0(C, M) \geq h^0(C, S_C^\vee) = 6$ we have $h^0(C, F) \geq 2$ and therefore $\deg(F) \geq 5$, i.e., $\deg(M) \leq 9$. Since C has no g_8^3 we must have $\deg(L) \geq 9$. Together this implies that $\deg(M) = 9$, i.e., $L \in W_9^3(C)$ and $F \in W_5^1(C)$. This means S_C^\vee sits in the exact sequence

$$0 \rightarrow F \rightarrow S_C^\vee \rightarrow Q_L \rightarrow 0.$$

But from Lemma 2.3 we also have

$$0 \rightarrow Q_L \rightarrow S_C^\vee \rightarrow F \rightarrow 0.$$

So we have the composition $\varphi: Q_L \xrightarrow{\beta} S_C^\vee \xrightarrow{\alpha} Q_L$ which is nonzero since $\ker(\alpha) = F$. Because Q_L is stable, φ is a homothety and in particular invertible. Then $\beta \circ \varphi^{-1}$ induces a splitting $S_C^\vee = F \oplus Q_L$. But since we have

$$0 \rightarrow S_C \rightarrow H^0(C, E_C) \otimes \mathcal{O}_C \rightarrow E_C \rightarrow 0$$

and E_C is nonsplit, this is a contradiction. ■

2.5 Stability of the normal bundle

We have established that S_C^\vee is stable. In order to prove the same for $\bigwedge^2 S_C^\vee$, we need some heavy machinery.

Theorem 2.8 ([Kob82; Lü83; Don85; UY86]). *Let E be a vector bundle on a complex smooth projective variety. If E is Hermite–Einstein then it is polystable. If E is stable then it is an Hermite–Einstein bundle.*

Corollary 2.9. *If E is a stable vector bundle on a complex smooth projective variety, then $\bigwedge^q E$ is polystable. Analogous statements hold for $\text{Sym}^q E$ and tensor products of stable vector bundles.*

Proof. E is Hermite–Einstein by Theorem 2.8, hence $\bigwedge^q E$ is as well (see e.g. [Lü83]). The result then follows by using Theorem 2.8 again. ■

Note that since we are on a curve, we could just as well use the Narasimhan–Seshadri theorem [NS65], which is twenty years earlier and less involved. With the eventual study of normal bundles of polarized K3 surfaces in mind, however, we have cited here the more general result.

Hence we know $\mathcal{N}_{C/\mathbb{P}^7}(-1) = \bigwedge^2 S_C^\vee$ is polystable and we are left with proving that it is indecomposable. We do this by excluding all possible ranks of bundles that could appear in a splitting.

Observe first that the sequence

$$0 \rightarrow Q_\eta \rightarrow S_C^\vee \rightarrow \zeta \rightarrow 0$$

from Lemma 2.3 for $\zeta \in W_5^1(C)$ and η the Serre dual, leads to

$$0 \rightarrow Q_\eta^\vee \otimes \eta \rightarrow \bigwedge^2 S_C^\vee \rightarrow Q_\eta \otimes \zeta \rightarrow 0. \quad (2.3)$$

This exact sequence will be fundamental in the proof. We will also need the following two facts:

Lemma 2.10. *Let $\xi \in W_5^1(C)$. Then $H^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \xi^{-1}) = 0$.*

Proof. Recall that $\mathcal{N}_{C/\mathbb{P}^7}(-1) = \mathcal{N}_{C/\mathbb{P}^7}^\vee(2)$. We have the standard identification

$$H^0(C, \mathcal{N}_{C/\mathbb{P}^7}^\vee(2)) = I_2(K_C)$$

i.e., the global sections of $\mathcal{N}_{C/\mathbb{P}^7}^\vee(2)$ correspond to the quadrics in \mathbb{P}^7 containing the canonical curve. If $\mu = K_C \otimes \xi^{-1}$ then $H^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \xi^{-1}) \neq 0$ implies that the multiplication map

$$\mathrm{Sym}^2 H^0(C, \mu) \rightarrow H^0(C, \mu^{\otimes 2})$$

is not injective. But $\mu \in W_9^3(C)$ and if the image under the induced embedding were contained in a quadric surface X , then a ruling of X would induce a \mathfrak{g}_4^1 on C . By assumption $W_4^1(C) = \emptyset$, so this is a contradiction. ■

Lemma 2.11. *Every sub-line bundle L of Q_η has degree at most 1. Every subbundle of Q_η of rank 2 has degree at most 4.*

Proof. Consider an exact sequence

$$0 \rightarrow L \rightarrow Q_\eta \rightarrow G \rightarrow 0$$

with L a line bundle. By increasing the degree of L , if needed, we may assume that G is a vector bundle as well. Then by Lemma 2.6 we have $h^0(C, \det(G)) \geq 3$ and since $W_7^2(C) = \emptyset$ we must have $\deg(G) \geq 8$. Hence $\deg(L) \leq 1$.

If $0 \rightarrow G \rightarrow Q_\eta$ is a rank 2 subbundle of maximal degree then the quotient is a globally generated line bundle L , hence $\deg(L) \geq 5$. This implies $\deg(G) \leq 4$. ■

Theorem 2.12. *The normal bundle of a genus 8 curve with $W_7^2(C) = \emptyset$ is stable.*

Proof. We have to show $\bigwedge^2 S_C^\vee$ does not decompose into a direct sum of stable bundles of slope $\mu(\bigwedge^2 S_C^\vee) = 7$. Assume that we can write $\bigwedge^2 S_C^\vee = F \oplus G$ where F is stable of slope $\mu(F) = 7$. We fix some notation. Let $i_F: F \hookrightarrow F \oplus G$ be the inclusion and denote by q the surjective map $\bigwedge^2 S_C^\vee \rightarrow Q_\eta \otimes \zeta$ from equation (2.3). Let

$$\varphi = q \circ i_F: F \rightarrow Q_\eta \otimes \zeta.$$

We claim that $\mathrm{rk}(\varphi)$ cannot be 1.

Assume the contrary. Then $B = \mathrm{im}(\varphi)$ is a line bundle and we have exact sequences

$$0 \rightarrow \ker(\varphi) \rightarrow F \rightarrow B \rightarrow 0$$

and

$$0 \rightarrow B \rightarrow Q_\eta \otimes \zeta \rightarrow \mathrm{coker}(\varphi) \rightarrow 0.$$

Lemma 2.11 implies that every sub-line bundle of $Q_\eta \otimes \zeta$ has degree at most 6, hence $\deg(B) \leq 6$ as well. But F is stable and has slope $\mu(F) = 7$, hence $\deg(B) \geq 7$ (we even have $\deg(B) \geq 8$ if $\mathrm{rk}(F) \geq 2$). This is impossible.

Now we deal with the case $\mathrm{rk}(\varphi) = 0$, i.e., $\varphi = 0$. This means that F is contained in the kernel of q , which is $Q_\eta^\vee \otimes \eta$, that is we get $0 \rightarrow F \rightarrow Q_\eta^\vee \otimes \eta$. This is impossible since $\mu(F) = 7$ and $\mu(Q_\eta^\vee \otimes \eta) = 6$, but $Q_\eta^\vee \otimes \eta$ is stable.

We now analyze all possible cases for the rank of F . By swapping F and G , if necessary, we may assume F is stable of rank 1, 2 or 3. The case of $\mathrm{rk}(F) = 1$ can already be excluded, since then $\mathrm{rk}(\varphi) \leq 1$.

If $\text{rk}(F) = 2$ then we must have $\text{rk}(\varphi) = 2$, i.e.,

$$0 \rightarrow F \rightarrow Q_\eta \otimes \zeta \rightarrow B \rightarrow 0$$

where B is a line bundle, since every rank 2 subbundle of $Q_\eta \otimes \zeta$ has slope at most 7. Tensoring by ζ^{-1} we obtain

$$0 \rightarrow F \otimes \zeta^{-1} \rightarrow Q_\eta \rightarrow B \otimes \zeta^{-1} \rightarrow 0$$

with $\deg(B \otimes \zeta^{-1}) = 5$, hence $h^0(C, B \otimes \zeta^{-1}) = 2$. Comparing this to $h^0(C, Q_\eta)$ we find $h^0(C, F \otimes \zeta^{-1}) \geq 2$ and hence $h^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \zeta^{-1}) \geq 2$ in contradiction of Lemma 2.10.

Finally, consider the case $\text{rk}(F) = 3$. If $\text{rk}(\varphi) = 2$ then we have the sequences

$$0 \rightarrow \ker(\varphi) \rightarrow F \rightarrow \text{im}(\varphi) \rightarrow 0, \quad 0 \rightarrow \text{im}(\varphi) \rightarrow Q_\eta \otimes \zeta \rightarrow \text{coker}(\varphi) \rightarrow 0$$

which imply $\mu(\text{im}(\varphi)) > 7$ and $\mu(\text{im}(\varphi)) \leq 7$, respectively (Q_η does not contain a rank 2 subbundle of degree at least 5). The last remaining possibility is then $\text{rk}(\varphi) = 3$, i.e., $0 \rightarrow F \rightarrow Q_\eta \otimes \zeta$. The quotient is a torsion sheaf of length 3. Tensoring the sequence by ζ^{-1} and comparing global sections, we again find $h^0(C, F \otimes \zeta^{-1}) \geq 1$ in contradiction to $h^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \zeta^{-1}) = 0$. \blacksquare

3 Polystability of the normal bundle on curves with a g_7^2

Ide and Mukai describe canonical models of genus 8 curves having a g_7^2 in [IM03]. As long as $W_4^1(C) = \emptyset$, the scheme $W_7^2(C)$ has length exactly 2. In the general case, there are precisely two non-autoresidual g_7^2 , say $W_7^2(C) = \{\alpha, \beta\}$ with $\alpha \neq \beta$. We then get a map $\varphi: C \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ induced by the linear system $|\alpha| \times |\beta|$. It is shown in [IM03] that φ is an embedding and its image a complete intersection of divisors of type $(1, 2)$, $(2, 1)$ and $(1, 1)$.

Let \mathcal{S} be the image of the Segre embedding $\sigma: \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^8$. Then the canonical model of C lies in $\sigma(\mathbb{P}^2 \times \mathbb{P}^2)$ intersected with a hyperplane $H = \mathbb{P}^7$. We let $W = \mathcal{S} \cap H$. So inside W , C is the transversal intersection of a divisor $D_{1,2}$ of type $(1, 2)$ and one divisor $D_{2,1}$ of type $(2, 1)$. We have the exact sequence

$$0 \rightarrow \mathcal{N}_{C/W} \rightarrow \mathcal{N}_{C/\mathbb{P}^7} \rightarrow \mathcal{N}_{W/\mathbb{P}^7}|_C \rightarrow 0.$$

Since C is a complete intersection in W , the normal bundle is just

$$\mathcal{N}_{C/W} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 2)|_C \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2, 1)|_C = \omega_C \otimes (\alpha \oplus \beta)$$

or equivalently $\mathcal{N}_{C/W}(-1) = \alpha \oplus \beta$ under the canonical embedding with $\mathcal{O}_C(1) = \omega_C$. We immediately get that $\mathcal{N}_{C/\mathbb{P}^7}$ is not stable, since it contains $\alpha \oplus \beta$ as a subbundle.

The following fact seems to be well-known and follows from the more general description of tangent bundles of flag varieties:

Lemma 3.1. *The normal bundle of \mathcal{S} in \mathbb{P}^8 is*

$$\mathcal{K}_{(1,0)} \otimes \mathcal{K}_{(0,1)} \otimes \mathcal{O}_{\mathcal{S}}(1,1)$$

where $\mathcal{K}_{(i,j)}$ is the kernel of the evaluation map $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(i,j)) \otimes \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_{\mathcal{S}}(i,j)$.

Using $\mathcal{N}_{W/H} = \mathcal{N}_{\mathcal{S}/\mathbb{P}^8}|_W$ and pulling this back to C we obtain that $\mathcal{N}_{W/\mathbb{P}^7}|_C(-1)$ is equal to $Q_{\alpha} \otimes Q_{\beta}$. Here Q_{α} and Q_{β} are the duals of the kernel bundles associated to α and β , respectively. Hence $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ sits in the exact sequence

$$0 \rightarrow \alpha \oplus \beta \rightarrow \mathcal{N}_{C/\mathbb{P}^7}(-1) \rightarrow Q_{\alpha} \otimes Q_{\beta} \rightarrow 0$$

and because $\mathcal{N}_{C/\mathbb{P}^7}$ is self-dual up to twist, this sequence splits. Therefore

$$\mathcal{N}_{C/\mathbb{P}^7}(-1) = \alpha \oplus (Q_{\alpha} \otimes Q_{\beta}) \oplus \beta. \quad (3.1)$$

To prove that this is in fact polystable, we will show the stability of $Q_{\alpha} \otimes Q_{\beta}$. Some preliminary results will be needed.

Lemma 3.2. *Q_{β} is stable and $h^0(C, Q_{\beta}) = 3$. The maximal degree of a line subbundle of Q_{β} is 2. Furthermore,*

$$H^0(C, Q_{\beta} \otimes \beta^{-1}) = H^0(C, Q_{\beta} \otimes \alpha^{-1}) = 0.$$

Analogous statements hold for Q_{α} .

Proof. Stability is shown as in Lemma 2.5. Now Q_{β} sits in the exact sequence

$$0 \rightarrow \mathcal{O}_C(a+b) \rightarrow Q_{\beta} \rightarrow \beta \otimes \mathcal{O}_C(-a-b) \rightarrow 0$$

where $a, b \in C$ are map to a singular point of the degree 7 plane model induced by β , hence $\beta \otimes \mathcal{O}_C(-a-b)$ is a \mathfrak{g}_5^1 . This shows $h^0(C, Q_{\beta}) = 3$ and $\deg(L) \leq 2$ for every line subbundle L of Q_{β} . To see the last claim, tensor with β^{-1} or α^{-1} and take global sections. \blacksquare

Remark 3.3. Since $h^0(C, \mathcal{N}_{C/\mathbb{P}^7}) = 15$, equation (3.1) yields $h^0(C, Q_{\alpha} \otimes Q_{\beta}) = 9$ as a consequence. One can also proceed similarly to [Muk10], section 5, to prove this result directly.

Lemma 3.4. *$Q_{\alpha} \otimes Q_{\beta}$ is stable.*

Proof. As a tensor product of stable bundles we already know $Q_{\alpha} \otimes Q_{\beta}$ is polystable (see Theorem 2.8) of slope $\mu(Q_{\alpha} \otimes Q_{\beta}) = 7$.

Consider the exact sequence

$$0 \rightarrow Q_{\alpha}(a+b) \xrightarrow{p} Q_{\alpha} \otimes Q_{\beta} \xrightarrow{q} Q_{\alpha} \otimes \xi \rightarrow 0$$

where $\beta(-a-b) = \xi \in W_5^1(C)$ and assume there exists a line bundle direct summand L of $Q_{\alpha} \otimes Q_{\beta}$. The induced map $L \rightarrow Q_{\alpha} \otimes \xi$ cannot be zero, since otherwise we get

$L \hookrightarrow \ker(q) = Q_\alpha(a+b)$, contradicting the stability of $Q_\alpha(a+b)$. So $L \rightarrow Q_\alpha \otimes \xi$ is injective, hence we obtain

$$0 \rightarrow L \rightarrow Q_\alpha \otimes \xi \rightarrow B \rightarrow 0$$

where B is a line bundle of degree 10. Looking at global sections, $h^0(C, L \otimes \xi^{-1}) = 1$ and hence ξ is a subbundle of L . But L , as a direct summand of the globally generated bundle $Q_\alpha \otimes Q_\beta$, is itself globally generated, so $L \in W_7^2(C)$. If $L = \alpha$ then

$$H^0(C, Q_\alpha \otimes Q_\beta \otimes L^{-1}) = H^0(C, Q_\alpha^\vee \otimes Q_\beta) = \text{Hom}(Q_\alpha, Q_\beta) = 0$$

by the stability of Q_α and Q_β and the fact that Q_α and Q_β are not isomorphic. The same statement holds if $L = \beta$. So $Q_\alpha \otimes Q_\beta$ does not contain a line bundle as direct summand.

Assume now that $Q_\alpha \otimes Q_\beta = F \oplus G$ where F and G are two rank 2 bundles. Consider again the sequence

$$0 \rightarrow Q_\alpha(a+b) \xrightarrow{p} F \oplus G \xrightarrow{q} Q_\alpha \otimes \xi \rightarrow 0$$

and the induced maps $\varphi: Q_\alpha(a+b) \rightarrow F$ and $\psi: Q_\alpha(a+b) \rightarrow G$. If $\varphi = 0$ this would induce a surjection $Q_\alpha \otimes \xi \rightarrow F \rightarrow 0$, which is impossible. If the rank of φ was 1 then we would get

$$Q_\alpha(a+b) \rightarrow \text{im}(\varphi) \rightarrow 0, \quad 0 \rightarrow \text{im}(\varphi) \rightarrow F$$

and hence $\deg(\text{im}(\varphi)) \geq 7$ and $\deg(\text{im}(\varphi)) < 7$, respectively. So the only possibility is that φ is injective. The same reasoning applies to ψ , hence $Q_\alpha(a+b) \hookrightarrow F$ and $Q_\alpha(a+b) \hookrightarrow G$. This implies $h^0(C, F(-a)) \geq 3$, $h^0(C, G(-a)) \geq 3$, i.e., $h^0(C, Q_\alpha \otimes Q_\beta \otimes \mathcal{O}_C(-a)) \geq 6$. But $Q_\alpha \otimes Q_\beta$ is globally generated of rank 4, so

$$h^0(C, Q_\alpha \otimes Q_\beta \otimes \mathcal{O}_C(-a)) = h^0(C, Q_\alpha \otimes Q_\beta) - 4 = 5$$

which is again a contradiction. ■

Remark 3.5. Since we have $E_C = \alpha \oplus \beta$ for the Mukai bundle on curves with a g_7^2 but no g_4^1 we can still form the exact sequence

$$0 \rightarrow S_C \rightarrow H^0(C, E_C) \otimes \mathcal{O}_C \rightarrow E_C \rightarrow 0,$$

even though it does not come from a tautological exact sequence over some Grassmannian, and then $S_C^\vee = Q_\alpha \oplus Q_\beta$. We also get

$$\bigwedge^2 S_C^\vee = \alpha \oplus (Q_\alpha \otimes Q_\beta) \oplus \beta$$

which is exactly the normal bundle $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ of the canonical embedding of C . So the normal bundle has intrinsically the same description (in terms of the Mukai bundle) as on the general curve.

A The normal bundle in genus 6

Every genus 6 curve is tetragonal, hence by [AFO16], Proposition 3.2, we know that the normal bundles of canonical genus 6 curves are never semistable. The proof in loc. cit. uses a rational scroll induced by the tetragonal pencil. In the spirit of the previous sections, we are going to give an alternative proof for the general curve using Mukai's description of genus 6 curves as quadric sections of Grassmannians.

By [Muk93], section 5, a canonical genus 6 curve C is an intersection of $G = G(2, 5) \subseteq \mathbb{P}^9$ with a 4-dimensional quadric if and only if $W_4^1(C)$ is finite, i.e., if C is not trigonal, bielliptic or a plane quintic. The intersection of G with a $\mathbb{P}^5 = H \subseteq \mathbb{P}^9$ is a del Pezzo surface X and then C is a quadric hypersurface section of S .

The inclusion $C \hookrightarrow G$ in the Grassmannian naturally leads to a destabilizing sequence for the normal bundle $\mathcal{N}_{C/\mathbb{P}^5}$. Let S and Q be the tautological bundle and the universal quotient bundle of G , respectively. We then have (as in section 2.3)

$$\mathcal{N}_{X/\mathbb{P}^5} \cong \mathcal{N}_{G(2,5)/\mathbb{P}^9}|_X \cong (\wedge^2 S^\vee) \otimes \det Q.$$

Furthermore, the exact sequence of normal bundles

$$0 \rightarrow \mathcal{N}_{C/X} \rightarrow \mathcal{N}_{C/\mathbb{P}^5} \rightarrow \mathcal{N}_{X/\mathbb{P}^5}|_C \rightarrow 0$$

is split, i.e.,

$$\mathcal{N}_{C/\mathbb{P}^5} \cong \mathcal{N}_{C/X} \oplus \mathcal{N}_{X/\mathbb{P}^5}|_C = \left(\bigwedge^2 S_C^\vee \right) \otimes \omega_C.$$

Here S_C denotes the restriction of S to C . Since S_C^\vee is of rank 3 we have

$$\bigwedge^2 S_C^\vee \cong S_C \otimes \det(S_C^\vee) \cong S_C \otimes \omega_C$$

implying $\mathcal{N}_{X/\mathbb{P}^5}|_C = S_C \otimes \omega_C^{\otimes 2}$. Now this is a vector bundle of slope $(\deg \omega_C^{\otimes 5})/3 = 50/3$ and it is a direct summand of $\mathcal{N}_{C/\mathbb{P}^5}$, which has slope $(\deg \omega_C^{\otimes 7})/4 = 70/4$. Since $50/3 < 70/4$, the normal bundle is unstable.

B The genus 9 case

The next case to consider in order to gain information on Conjecture 1.1 would be canonical curves of genus 9. For the general (i.e. non-pentagonal) curve we again have a Grassmannian embedding, this time into the symplectic Grassmannian $\text{SpG}(3, 6)$

$$C \hookrightarrow \text{SpG}(3, 6) \subseteq \mathbb{P}^{13}.$$

The embedding is again given by a uniquely defined Mukai bundle E_C , here of rank 3. The details can be found in [Muk10]. C can be recovered as a transversal intersection of $\text{SpG}(3, 6)$ with an 8-plane.

We may again use this description to obtain more information about the normal bundle of the canonical embedding $C \hookrightarrow \mathbb{P}^8$. However, the situation seems to be considerably more complicated and our methods run out of steam before we can fully prove stability.

Let us consider what information about $\mathcal{N}_{C/\mathbb{P}^8}$ we can get from the Grassmannian embedding. $\mathrm{SpG}(3,6)$ is a subvariety of the Grassmannian $G = G(3,6)$ of 3-planes in $H^0(C, E_C)$. On G we have the universal exact sequence

$$0 \rightarrow S_G \rightarrow V \otimes \mathcal{O}_G \rightarrow Q_G \rightarrow 0$$

where Q_G is the universal quotient bundle and S_G is the tautological bundle. The sequence restricts to $X = \mathrm{SpG}(3,6)$ and on X we can identify S_X with Q_X^\vee :

$$0 \rightarrow Q_X^\vee \rightarrow V \otimes \mathcal{O}_X \rightarrow Q_X \rightarrow 0.$$

X is embedded in \mathbb{P}^{13} by a restricted Plücker embedding.

Lemma B.1 ([Muk10, Section 2]). *The normal bundle of $\mathrm{SpG}(3,6) \subseteq \mathbb{P}^{13}$ sits in the exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{N}_{X/\mathbb{P}^{13}}^\vee(2) \rightarrow \mathrm{Sym}^2 Q_X \rightarrow 0$$

where $\mathcal{O}_X(1) = \det Q_X$.

Since C is the intersection $C = \mathbb{P}^8 \cap X$, we get that the normal bundle $\mathcal{N}_{C/\mathbb{P}^8}$ of the canonical embedding is equal to the normal bundle $\mathcal{N}_{X/\mathbb{P}^{13}}$ of X , restricted to C . Furthermore, the restriction of Q_X to C is the Mukai bundle E_C . Thus we obtain the following exact sequence for the normal bundle of C .

Lemma B.2. *The twisted conormal bundle of $C \subseteq \mathbb{P}^8$ in its canonical embedding sits in the exact sequence*

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{N}_{C/\mathbb{P}^8}^\vee(2) \rightarrow \mathrm{Sym}^2 E_C \rightarrow 0. \quad (\text{B.1})$$

Observe that the universal exact sequence of X restricts to

$$0 \rightarrow E_C^\vee \rightarrow H^0(C, E_C) \rightarrow E_C \rightarrow 0$$

on C .

Already it is a non-trivial calculation to show that

$$\mathrm{Sym}^2 H^0(C, E_C) \rightarrow H^0(C, \mathrm{Sym}^2 E_C)$$

is an isomorphism and hence $h^0(C, \mathrm{Sym}^2 E_C) = 21$. Having established this, we can make use of representation theory (see [FH91, Exercise 15.36]) to identify the global sections of $\mathrm{Sym}^2 E_C$ with the space $I_2(K_C)$ of quadrics containing the canonical embedding of C .

One can then use the exact sequence (B.1) to prove that $\mathcal{N}_{C/\mathbb{P}^8}$ has no destabilizing quotients of rank 1 or 2. Furthermore, one can obtain bounds for the degrees of quotient bundles of other ranks. The calculations are similar to the genus 8 case, just more lengthy. However, we fear that these methods may not be potent enough to establish stability with respect to quotient bundles of all ranks.

As a final note we present here some easy facts that might prove useful in tackling the question of stability of the normal bundle of a general canonical curve for arbitrary g . First we establish normal generation for twists of the normal and the conormal bundle.

Lemma B.3. $\mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2)$ is globally generated for every canonical curve with Clifford index $\text{Cliff}(C) \geq 2$.

Proof. First note that $\text{Cliff}(C) \geq 2$ implies that C is scheme-theoretically cut out by quadrics. Now observe that

$$H^0(C, \mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2)) = I_2(K_C) \oplus \ker \psi$$

where $I_2(K_C)$ is the space of quadrics containing C , and by ψ we denote the Wahl map $\wedge^2 H^0(C, K_C) \rightarrow H^0(C, K_C^{\otimes 3})$.

Consider the map

$$f: I_2(K_C) \otimes \mathcal{O}_C \rightarrow \mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2)|_p$$

to the fiber at a point p . Its kernel are quadrics in \mathbb{P}^{g-1} vanishing on the curve, but also vanishing at p to order 2. Hence it is naturally identified with the kernel of the map

$$\text{Sym}^2 H^0(C, K_C(-p)) \rightarrow H^0(C, K_C^{\otimes 2} \otimes \mathcal{O}_C(-2p)).$$

Since this map is surjective ([GL86]) we can calculate the dimension of the kernel and hence the image of f . We get $\dim \text{im}(f) = g - 2$, so f is surjective. \blacksquare

Lemma B.4. $\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1)$ is globally generated for all canonical curves of every genus g .

Proof. We imitate the proof in [AFO16, p. 12]. Via Serre duality, the claim is equivalent to the equality $h^0(C, \mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2) \otimes \mathcal{O}_C(p)) = h^0(C, \mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2))$ for every $p \in C$. Consider the pullback of the ideal sequence of C in \mathbb{P}^{g-1} to C and its twist by $K_C^{\otimes 2}(p)$:

$$0 \rightarrow \mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2) \otimes \mathcal{O}_C(p) \rightarrow M_{K_C} \otimes K_C(p) \rightarrow K_C^{\otimes 3}(p) \rightarrow 0.$$

So we obtain that $H^0(C, \mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(2) \otimes \mathcal{O}_C(p))$ is the kernel of the twisted Wahl map

$$H^0(C, M_{K_C} \otimes K_C(p)) \rightarrow H^0(C, K_C^{\otimes 3}(p)).$$

But because Q_{K_C} is globally generated, the dimension of $H^0(C, M_{K_C} \otimes K_C(p))$ is equal to that of $H^0(C, M_{K_C} \otimes K_C)$. \blacksquare

This has an interesting consequence.

Lemma B.5. For every $r_0 > 0$ there is an integer g_0 such that the normal bundle $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ of a general canonical curve of every genus $g \geq g_0$ has no destabilizing quotient bundle of rank $r \leq r_0$.

Proof. Let $\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1) \rightarrow F \rightarrow 0$ be a destabilizing quotient of rank r . Then F is globally generated and has slope

$$\mu(F) \leq \mu(\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1)) = 6 \left(\frac{g-1}{g-2} \right)$$

We may assume $g \geq 8$. Then in particular, $d = \deg(F) \leq 7r$. By Lemma 2.6 we must have $h^0(C, \det(F)) \geq r + 1$. But if g is large enough, then $\rho(g, r, d) < 0$. We conclude that for these g no such F exists. \blacksquare

Finally, let us note that in general it seems hard to decide when precisely the normal bundle of an arbitrary (non-general) curve C is unstable. Not having a tetragonal pencil does not seem to be the right condition:

Example B.6. The normal bundle of a canonical curve can be unstable even if the curve is not tetragonal. Consider a curve of genus 9 with a $\mathfrak{g}_9^3 =: L$ but no \mathfrak{g}_4^1 . Such curves are explicitly constructed in M. Sagraloff's Ph.D. thesis [Sag06]. We have $\rho(9, 3, 9) = -3$ as well as $\rho(9, 1, 4) = -3$, but the two Brill–Noether loci $\mathcal{M}_{9,1}^4$ and $\mathcal{M}_{9,3}^9$ are not equal.

Consider the Serre dual $A = \omega_C \otimes L^{-1}$ of L , which is a \mathfrak{g}_7^2 . By using L and A , we obtain an embedding of C into $\mathbb{P}^2 \times \mathbb{P}^3$. By composing this with the Segre embedding, we recover the canonical image of C . Hence we get a quotient $Q_L \otimes Q_A$ of $\mathcal{N}_{C/\mathbb{P}^8}(-1)$ of slope $\mu(Q_A \otimes Q_L) = 13/2$. However, note that the normal bundle has slope $\mu(\mathcal{N}_{C/\mathbb{P}^8}(-1)) = 48/7$, which is strictly bigger.

References

- [AFO16] M. Aprodu, G. Farkas, and A. Ortega. “Restricted Lazarsfeld–Mukai bundles and canonical curves”. *Development of moduli theory – Kyoto 2013*. Ed. by O. Fujino, S. Kondō, A. Moriwaki, M.-H. Saito, and K. Yoshioka. Vol. 69. Advanced studies in pure mathematics. Mathematical Society of Japan, 2016, pp. 303–322 (cit. on pp. 1, 2, 14, 16).
- [Don85] S. K. Donaldson. “Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles”. *Proc. London Math. Soc.* 50 (1985), 1–26 (cit. on p. 9).
- [EL92] L. Ein and R. Lazarsfeld. “Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves”. *Complex projective geometry*. Ed. by G. Ellingsrud, C. Peskine, G. Sacchiero, and S. A. Stromme. Vol. 179. London Mathematical Society Lecture Note Series. Cambridge University Press, 1992, pp. 149–156 (cit. on p. 2).
- [ES03] D. Eisenbud and F.-O. Schreyer. “Resultants and Chow forms via exterior syzygies (with an appendix by J. Weyman)”. *J. Amer. Math. Soc.* 16 (2003), 537–579 (cit. on p. 4).
- [ES12] F. Eusen and F.-O. Schreyer. “A remark on a conjecture of Paranjape and Ramanan”. *Geometry and Arithmetic*. Ed. by C. Faber, G. Farkas, and R. de Jong. Eur. Math. Soc., 2012, pp. 113–123 (cit. on pp. 1, 7).
- [EV82] D. Eisenbud and A. van de Ven. “On the variety of smooth rational space curves with given degree and normal bundle”. *Invent. Math.* 67 (1982), 89–100 (cit. on p. 2).
- [Far09] G. Farkas. “Koszul divisors on moduli spaces of curves”. *Amer. J. Math.* 131 (2009), 819–867 (cit. on p. 4).
- [FH91] W. Fulton and J. Harris. *Representation theory*. New York: Springer, 1991 (cit. on p. 15).

- [FO12] G. Farkas and A. Ortega. “Higher rank Brill–Noether theory on sections of K3 surfaces”. *Int. J. Math.* 23 (2012) (cit. on p. 4).
- [GL86] M. Green and R. Lazarsfeld. “On the projective normality of complete linear series on an algebraic curve”. *Invent. Math.* 83 (1986), 73–90 (cit. on p. 16).
- [HS83] K. Hulek and G. Sacchiero. “On the normal bundle of elliptic space curves”. *Arch. Math.* 40 (1983), 61–68 (cit. on p. 2).
- [IM03] M. Ide and S. Mukai. “Canonical curves of genus eight”. *Proc. Japan Acad.* 79 (2003), 59–64 (cit. on pp. 2, 11).
- [Kob82] S. Kobayashi. “Curvature and stability of vector bundles”. *Proc. Japan Acad. Ser. A Math. Sci.* 58 (1982), 158–162 (cit. on p. 9).
- [Laz86] R. Lazarsfeld. “Brill–Noether–Petri without degenerations”. *J. Differential Geom.* 23 (1986), 299–307 (cit. on p. 4).
- [Lü83] M. Lübke. “Stability of Einstein–Hermitian vector bundles”. *Manuscripta Math.* 42 (1983), 245–257 (cit. on p. 9).
- [Man98] L. Manivel. “Gaussian maps and plethysm”. *Algebraic geometry: papers presented for the EUROPROJ conferences in Catania and Barcelona*. Ed. by P. E. Newstead. New York: Marcel Dekker, Inc., 1998, pp. 91–117 (cit. on p. 5).
- [Muk10] S. Mukai. “Curves and symmetric spaces, II”. *Ann. of Math.* (2) 172 (2010), 1539–1558 (cit. on pp. 12, 14, 15).
- [Muk88] S. Mukai. “Curves, K3 surfaces and Fano 3-folds of genus ≤ 10 ”. *Algebraic Geometry and Commutative Algebra*. Tokyo: Kinokuniya, 1988, pp. 357–377 (cit. on p. 3).
- [Muk93] S. Mukai. “Curves and Grassmannians”. *Algebraic Geometry and Related Topics*. Boston: International Press, 1993, pp. 19–40 (cit. on pp. 2, 4, 6, 14).
- [NS65] M. S. Narasimhan and C. S. Seshadri. “Stable and unitary vector bundles on a compact Riemann surface”. *Ann. of Math.* (2) 82 (1965), 540–567 (cit. on p. 9).
- [PR88] K. Paranjape and S. Ramanan. “On the canonical ring of a curve”. *Algebraic geometry and commutative algebra, vol. II*. Ed. by H. Hijikata, H. Hironaka, M. Maruyama, H. Matsumura, M. Miyanishi, T. Oda, and K. Ueno. Orlando, Florida: Academic Press, 1988, pp. 503–516 (cit. on p. 8).
- [Sag06] M. Sagraloff. “Special linear series and syzygies of canonical curves of genus 9”. PhD thesis. Universität des Saarlandes, 2006 (cit. on p. 17).
- [UY86] K. Uhlenbeck and S. T. Yau. “On the existence of Hermitian–Yang–Mills connections in stable vector bundles”. *Comm. Pure Appl. Math.* 39 (1986), 257–293 (cit. on p. 9).
- [Voi02] C. Voisin. “Green’s generic syzygy conjecture for curves of even genus lying on a K3 surface”. *J. Eur. Math. Soc.* 4 (2002), 363–404 (cit. on p. 4).
- [Voi05] C. Voisin. “Green’s canonical syzygy conjecture for generic curves of odd genus”. *Compos. Math.* 141 (2005), 1163–1190 (cit. on p. 4).