Birational geometry of modular varieties

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Objects of study

We study the modular variety $\mathcal{R}_{g,\ell}$ whose points correspond to pairs $[C, \eta]$ where

- C is a curve of genus g
- η is a nontrivial point of order ℓ in the Jacobian of C.

Hence $\mathcal{R}_{g,\ell}$ is a $(\ell^{2g} - 1)$-fold étale cover of $\mathcal{M}_g$, the moduli space of curves of genus g:

$$\mathcal{R}_{g,\ell} \to \mathcal{M}_g, \quad [C, \eta] \mapsto [C]$$

We can also view $\mathcal{R}_{g,\ell}$ as parametrizing cyclic étale $(\ell : 1)$ covers $\widetilde{C} \to C$. Using this interpretation, we get an embedding of $\mathcal{R}_{g,\ell}$ into $\mathcal{M}_{\ell g-(\ell-1)}$ as the locus of curves having a fix-point free automorphism of order ℓ.

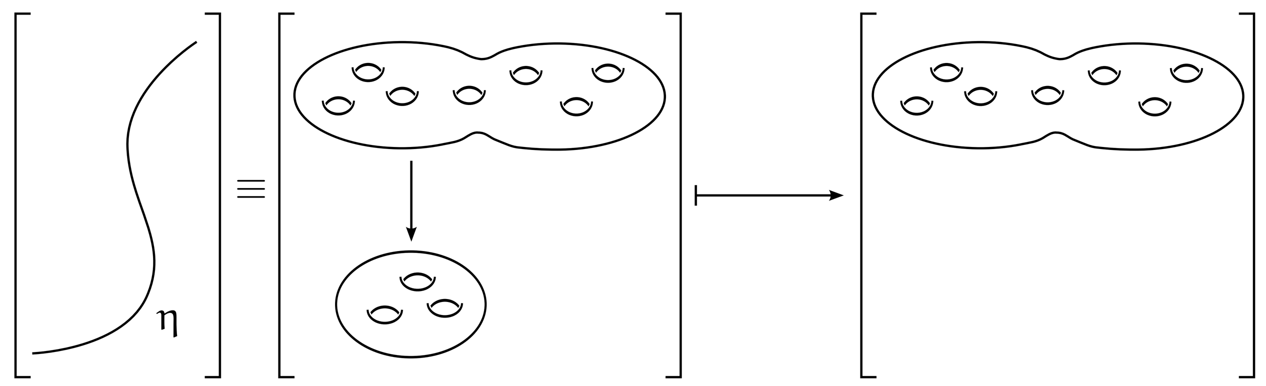

Fig. 1: The map $\mathcal{R}_{3,3} \to \mathcal{M}_7$ sending $[\widetilde{C} \to C]$ to $\widetilde{C}$

$\mathcal{R}_{g,\ell}$ can be seen as a higher genus analogue of the modular curve $Y_1(\ell)$ which parametrizes pairs of elliptic curves and an ℓ-torsion point. It is also a higher level generalization of the moduli space of Prym varieties $\mathcal{R}_g = \mathcal{R}_{g,2}$.

We are interested in the birational geometry of the compactification $\overline{\mathcal{R}}_{g,\ell}$ of $\mathcal{R}_{g,\ell}$. The following picture shows some degenerations.

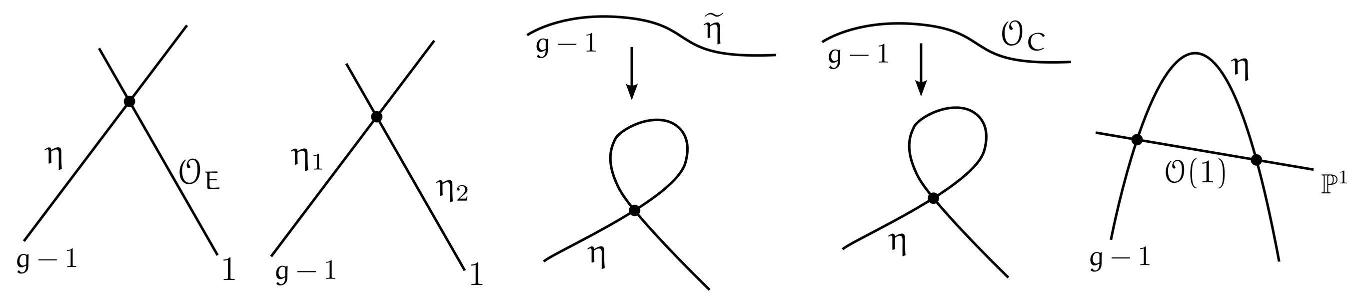

Fig. 2: Typical curves in the boundary of $\overline{\mathcal{R}}_{g,2}$

Guiding questions (Description of geometric loci)**.**

- List and understand subloci of $\overline{\mathcal{R}}_{g,\ell}$ defined in terms of η and the geometry of C.
- What is the Kodaira dimension of $\overline{\mathcal{R}}_{g,\ell}$?
- Understand properties of the locus of curves that are ℓ-fold étale covers. How general are they in their moduli?

Questions

Known geometric loci are largely defined in terms of special line bundles on curves. For certain triples (g, r, d) a general curve of genus g will carry a finite number of $\mathfrak{g}^r_d$, i.e. linear series of degree d and dimension r. We can then consider loci defined in terms of global sections of the **twists** $L \otimes \eta$ where L is such a $\mathfrak{g}^r_d$. We may ask when $H^1(L \otimes \eta) \neq 0$ or equivalently when $L \otimes \eta$ has more than the expected number of sections..

Question (Sections of twists)**.** Assume $d \geqslant g - 1$. What is the codimension of

$$Z_{g,\ell} = \{ [C, \eta] \in \mathcal{R}_{g,\ell} \mid \exists L \text{ a } \mathfrak{g}^r_d \text{ s.t. } H^1(C, L \otimes \eta) \neq 0 \}$$

in $\mathcal{R}_{g,\ell}$? The expected value is $d - g + 2$ but we only know codim $Z_{g,\ell} \geqslant 1$.

On boundary curves the question becomes harder since the limit linear series theory by Eisenbud and Harris does not tell us how the limit $\mathfrak{g}^r_d$ relate to their twists. Note that the only case where we exactly know what the bundles η look like are elliptic curves, and hyperelliptic curves in the case $\ell = 2$.

Question (Difference varieties)**.** For which curves and which i does the difference variety

$$C_i - C_i = \{ \mathcal{O}_C(D) \mid D \sim D_+ - D_- \text{ where } D_+, D_- \in C^i \}$$

contain an ℓ-torsion point? How many? All of them?

To prove statements about the codimension of such cycles, it is very useful to be able to specialize to particular curves and line bundles on them. The moduli space of $\mathfrak{G}^{r,(\ell)}_d$ of triples $[C, \eta, L]$ then needs to enjoy certain irreducibility properties.

Question (Varieties of linear series)**.** When is $\mathfrak{G}^{r,(\ell)}_d$ irreducible? If it is reducible, when is there a unique component dominating $\mathcal{M}_g$?

Lemma. Let $g \geqslant 3$ and the Brill–Noether number $\rho(g, r, d) = 0$. Then there is a unique component of $\mathfrak{G}^{r,(\ell)}_d$ that maps dominantly to $\mathcal{M}_g$. If $r \leqslant 2$ then $\mathfrak{G}^{2,(\ell)}_d$ is irreducible.

Answers

Theorem ([FL10], [Chi+13])**.** *$\overline{\mathcal{R}}_{g,2}$ is of general type for $g \geqslant 14$, $g \neq 15$. $\overline{\mathcal{R}}_{g,3}$ is of general type for $g \geqslant 12$.*

Our aim is to close the gap at $g = 15$ and $\ell = 2$, i.e. show that $\overline{\mathcal{R}}_{15,2}$ is of general type.

Strategy: A curve of genus 6 can be mapped to $\mathbb{P}^2$ as a 4-nodal sextic curve C by an $L = \mathfrak{g}^2_6$. For some curves there exists a plane conic Q that is tangent to C at every point of intersection.

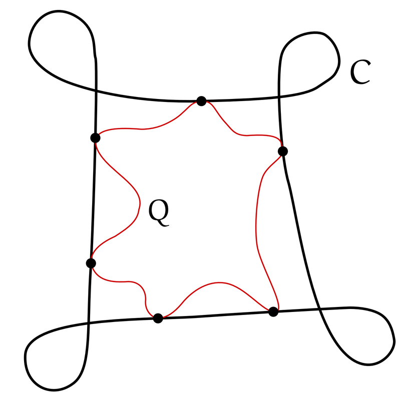

Fig. 3: A plane sextic C of genus 6 and a totally tangent quadric Q

The existence of such a conic is equivalent to the existence of a 2-torsion bundle η such that the multiplication map

$$\mathrm{Sym}^2 H^0(C, L \otimes \eta) \to H^0(C, L^{\otimes 2}) / \mathrm{Sym}^2 H^0(C, L)$$

is not bijective. This condition can be imitated for genus 15. Here the general curve has a finite number of $L = \mathfrak{g}^4_{16}$. The map

$$\mu_{[C,\eta,L]} \colon \mathrm{Sym}^2 H^0(C, L \otimes \eta) \to H^0(C, L^{\otimes 2}) / \mathrm{Sym}^2 H^0(C, L)$$

is an isomorphism for a general triple $[C, \eta, L]$. We study the divisor

$$D_{15} = \{ [C, \eta] \in \mathcal{R}_{15} \mid \exists L \text{ a } \mathfrak{g}^4_{16} \text{ s.t. } \mu_{[C,\eta,L]} \text{ not bijective} \}$$

and conditionally calculate the class of its closure, proving that the canonical class of $\overline{\mathcal{R}}_{15,2}$ is big.

Further reading

[Chi+13] A. Chiodo, D. Eisenbud, G. Farkas, and F.-O. Schreyer, "Syzygies of torsion bundles and the geometry of the level ℓ modular variety over $\overline{\mathcal{M}}_g$," *Invent. math.*, vol. 194, pp. 73–118, 2013.

[FL10] G. Farkas and K. Ludwig, "The Kodaira dimension of the moduli space of Prym varieties," *J. Eur. Math. Soc.*, vol. 12, pp. 755–795, 2010.