Divisors on moduli spaces of covers of curves

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Motivation

The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g is a central object in modern algebraic geometry. Interest in its global birational geometry began with Severi's work, who proved that $\mathcal{M}_g$ is unirational for $g \le 10$. On the other hand, Mumford–Harris and Eisenbud–Harris in the '80s showed that in fact $\mathcal{M}_g$ is of general type for $g \ge 24$. Since then a series of results have increased our knowledge, but for $17 \le g \le 21$ it is still unclear what the Kodaira dimension of $\mathcal{M}_g$ is.

In recent years interest has sparked in studying moduli spaces of curves with additional data, such as $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable curves with n marked points, $\overline{\mathcal{S}}_g$, the moduli space of spin curves, and $\overline{\mathcal{R}}_{g,\ell}$, the moduli space of curves with level ℓ structure.

These moduli spaces exhibit some very interesting birational geometry in their own right, but they are also studied to gain more insight into the fundamental questions about $\mathcal{M}_g$.

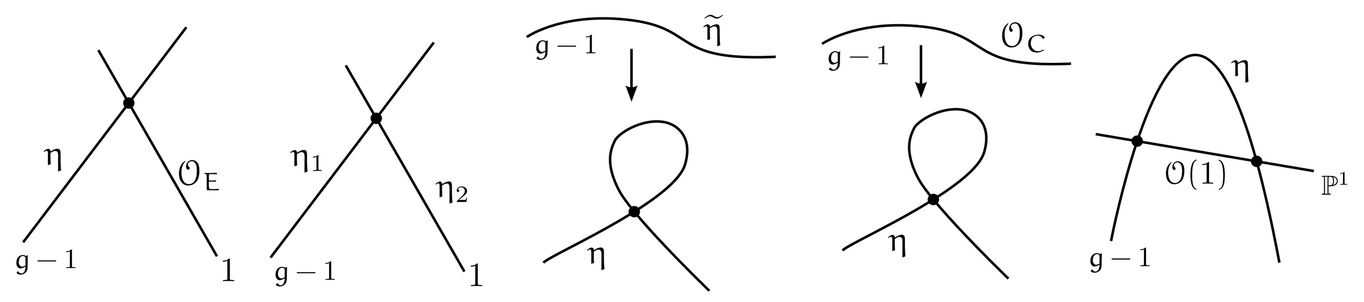
Objects of study

We study the modular variety $\mathcal{R}_{g,\ell}$ whose points correspond to pairs $[C,\eta]$ where

- C is a smooth curve of genus g
- η is a nontrivial point of order ℓ in the Jacobian of C.

Hence $\mathcal{R}_{g,\ell}$ is a finite cover of $\mathcal{M}_g$ by forgetting the torsion point. It can be seen as a higher genus analogue of the modular curve $Y_1(\ell)$ which parametrizes pairs of elliptic curves and an ℓ-torsion point. It is also a higher level generalization of the moduli space of Prym varieties $\mathcal{R}_{g,2}$.

Main interest. The birational geometry of the compactification $\overline{\mathcal{R}}_{g,\ell}$ of $\mathcal{R}_{g,\ell}$ by *Prym curves*. These are obtained from stable curves by blowing up irreducible nodes, plus an appropriate definition of torsion bundle. Here are some examples of general curves in the boundary of $\overline{\mathcal{R}}_{g,\ell}$:



Method. General type results of our moduli spaces can be proved by constructing an effective divisor $\mathcal{D}$ such that $K_{\overline{\mathcal{R}}_{g,\ell}} \equiv \mathcal{D} + \varepsilon\lambda$, where λ is the (big) Hodge class of $\overline{\mathcal{R}}_{g,\ell}$. Hence our main interest is in investigating divisors of $\overline{\mathcal{R}}_{g,\ell}$, usually defined in terms of η and the geometry of C.

State of the art. It is known that $\mathcal{R}_{g,2}$ is of general type for $g \ge 14$, $g \ne 15$ (see [2]). Furthermore we know that $\mathcal{R}_{g,3}$ is of general type for $g \ge 12$ and the Kodaira dimension of $\mathcal{R}_{11,3}$ is at least 19 (see [1]).

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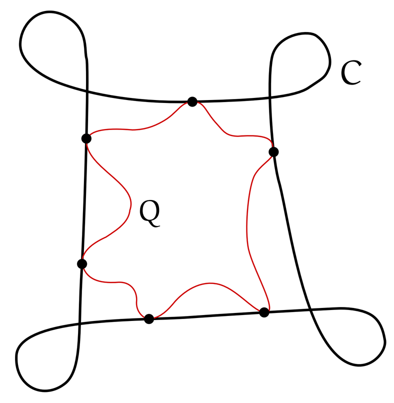
Results achieved

Theorem (—, 2015)**.** *The moduli space $\overline{\mathcal{R}}_{15,2}$ is of general type.*

We motivate the construction of the effective divisor $\mathcal{D}_{15}$ used in the proof. A general curve of genus 6 can be mapped to $\mathbb{P}^2$ as a 4-nodal sextic curve C by some line bundle L. We ask when there exists a plane conic Q that is tangent to C at every point of intersection:



The existence of such a conic is equivalent to the existence of a 2-torsion bundle η such that the multiplication map

$$\mu_{[C,\eta,L]}\colon \operatorname{Sym}^2 H^0(C, L\otimes\eta) \to H^0(C, L^{\otimes 2})/\operatorname{Sym}^2 H^0(C,L)$$

is not bijective. We now imitate this condition for genus 15. Here C has a degree 16 model in $\mathbb{P}^4$ induced by a bundle L. The analogous map $\mu_{[C,\eta,L]}$ is an isomorphism for a general triple $[C,\eta,L]$. We let

$$\mathcal{D}_{15} = \left\{[C,\eta] \in \mathcal{R}_{15} \;\middle|\; \exists L \text{ such that } \mu_{[C,\eta,L]} \text{ is not bijective}\right\}$$

Then we show that $\mathcal{D}_{15}$ is a divisor and calculate the class of its closure.

For the next level $\ell = 3$ we have the following result:

Theorem (—, 2016)**.** *The space $\overline{\mathcal{R}}_{8,3}$ is of general type.*

The proof rests on improving some results in [1] and studying the twist by η of the *Mukai bundle* E_C associated to a genus 8 curve C. The locus

$$\mathcal{D}_{8,3} = \left\{[C,\eta] \in \mathcal{R}_{8,3} \;\middle|\; H^0(C, E_C \otimes \eta) \ne 0\right\}$$

is the effective divisor in $\mathcal{R}_{8,3}$ which we use.

One is naturally led to ask what happens for $\mathcal{R}_{g,3}$ in the range $9 \le g \le 11$. The theorem suggests that these spaces should also be of general type.

References

[1] A. Chiodo, D. Eisenbud, G. Farkas, and F.-O. Schreyer. Syzygies of torsion bundles and the geometry of the level ℓ modular variety over $\overline{\mathcal{M}}_g$. *Invent. Math.* 194 (2013), 73–118.

[2] G. Farkas and K. Ludwig. The Kodaira dimension of the moduli space of Prym varieties. *J. Eur. Math. Soc.* 12 (2010), 755–795.

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