Moduli spaces of covers of curves

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Motivation

Most important object to study:

The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g, a central object in modern algebraic geometry. Its points parametrize isomorphism classes of curves of a fixed genus g.

Here is a smooth curve of genus 1, and one of genus 3:

What birational geometry tells us:

If $\overline{\mathcal{M}}_g$ is *unirational* for some g, i.e. there is a dominant rational map $\mathbb{P}^r \dashrightarrow \overline{\mathcal{M}}_g$, then we can write *almost all* genus g curves by varying the coefficients in some equations.

On the other end of the spectrum, $\overline{\mathcal{M}}_g$ might be *of general type*. Then these curves are as hard as possible to write down. For *almost all* curves the equations are rigid and very special.

History of the problem:

- Severi (1915): $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$.
- Severi's conjecture: $\overline{\mathcal{M}}_g$ is unirational for *all* g.
- Sernesi (1981): $\overline{\mathcal{M}}_{12}$ is unirational.
- Harris–Mumford (1982): $\overline{\mathcal{M}}_g$ is of general type for odd $g \geq 25$.
- Many authors: $\overline{\mathcal{M}}_g$ unirational for $g \leq 14$, of general type for $g \geq 22$.

Other moduli spaces:

Recently, there has been research in moduli spaces of curves with additional data. This data gives the problem more rigidity. For example:

- $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable curves with n marked points,
- $\overline{\mathcal{S}}_g$, the moduli space of spin curves, and
- $\overline{\mathcal{R}}_{g,\ell}$, the moduli space of curves with level ℓ structure.

These exhibit some very interesting birational geometry in their own right. They are also studied to gain more insight into the fundamental questions about $\overline{\mathcal{M}}_g$.

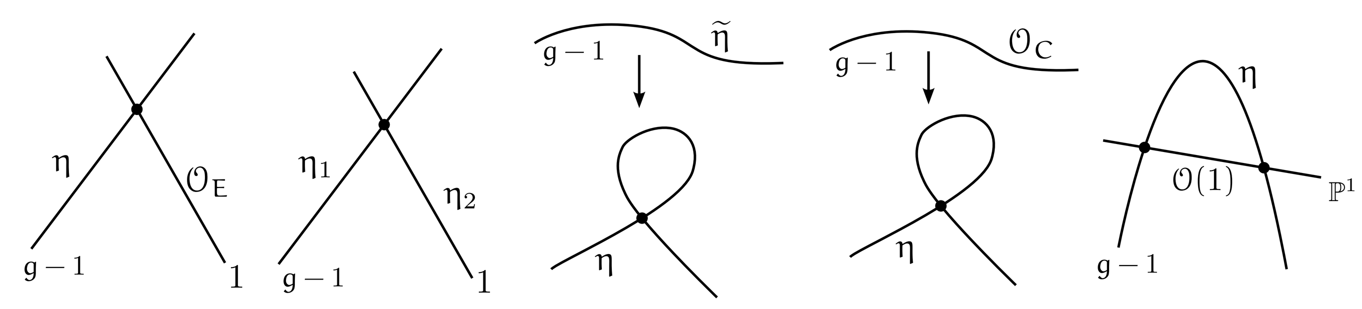
My favorite moduli spaces

My main interest is the modular variety $\mathcal{R}_{g,\ell}$ whose points correspond to pairs $[C, \eta]$, where

- C is a smooth curve of genus g
- η is a nontrivial point of order ℓ in the Jacobian of C.

Hence $\mathcal{R}_{g,\ell} \to \mathcal{M}_g$ is a finite cover by forgetting the order ℓ point. It can be seen as a higher genus analogue of the modular curve $Y_1(\ell)$ (parametrizing pairs of elliptic curves and an ℓ-torsion point).

Here are some examples of general curves in the boundary of $\overline{\mathcal{R}}_{g,\ell}$:



State of the art:

- $\mathcal{R}_{g,2}$ is of general type for $g \geq 14$, $g \neq 15$ (see [2]).
- $\mathcal{R}_{g,2}$ is unirational for $g \leq 7$.
- $\mathcal{R}_{g,3}$ is of general type for $g \geq 12$ (see [1]).

Method of proof:

General type results can be proved by constructing an effective divisor $\mathcal{D}$ such that $K_{\overline{\mathcal{R}}_{g,\ell}} \equiv \mathcal{D} + \varepsilon\lambda$, where λ is the (big) Hodge class of $\overline{\mathcal{R}}_{g,\ell}$ and $\varepsilon > 0$. Hence our main interest is in investigating divisors of $\overline{\mathcal{R}}_{g,\ell}$, usually defined in terms of η and the geometry of C.

Results

For the most classical case $\ell = 2$ (Prym varieties) we have:

Theorem (—, 2015). *The moduli space $\overline{\mathcal{R}}_{15,2}$ is of general type.*

For the next level $\ell = 3$ we have the following result:

Theorem (—, 2016). *The space $\overline{\mathcal{R}}_{8,3}$ is of general type.*

One is naturally led to ask what happens for $\mathcal{R}_{g,3}$ in the range $9 \leq g \leq 11$. The theorem suggests that these spaces should also be of general type.

References

[1] A. Chiodo, D. Eisenbud, G. Farkas, and F.-O. Schreyer. "Syzygies of torsion bundles and the geometry of the level ℓ modular variety over $\overline{\mathcal{M}}_g$". *Invent. Math.* 194 (2013), pp. 73–118.

[2] G. Farkas and K. Ludwig. "The Kodaira dimension of the moduli space of Prym varieties". *J. Eur. Math. Soc.* 12 (2010), pp. 755–795.