Decompositions of the diagonal and applications

Gregor Bruns

23 February, 2017

In a recent paper by K.-W. Lai the following was proved:

Theorem 1 ([Lai16])**.** *A generic* $X \in \mathcal{C}_{42}$ *has a degree* 13 *unirational parametrization.*

Lai also proved that $\mathcal{C}_{42}$ is uniruled. The goal of this talk is to explain a bit why results like these are a big deal. Uniruledness of $\mathcal{C}_{42}$ is actually easier to motivate, since there is Hassett's rational degree 2 cover $\mathcal{F}_{42} \dashrightarrow \mathcal{C}_{42}$. A lot of people these days are interested in the birational type of $\mathcal{F}_{2d}$ and hence these facts are interesting.

To see why someone would care about degree 13 unirational parametrizations, we have to introduce the whole story of *decompositions of the diagonal*. Let X be a smooth complex projective variety, $\dim(X) = n$.

Definition 2. A *Chow decomposition of the diagonal* of X is an equality of cycles in $CH^n(X \times X)$

$$N \cdot \Delta_X = N \cdot [x \times X] + Z$$

for some $N \in \mathbb{Z} \setminus \{0\}$, $x \in X(\mathbb{C})$ and some Z supported on $X \times D$ for a $D \subsetneq X$.

A *cohomological decomposition of the diagonal* of X is an equality of cohomology classes in $H^{2n}(X \times X, \mathbb{Z})$

$$N \cdot \Delta_X = N \cdot [x \times X] + Z$$

If $N = 1$, then the decomposition is said to be *integral* and we say that X *admits an integral decomposition of the diagonal* (either Chow or cohomological) if we can choose $N = 1$.

Definition 3. We denote by $N(X)$ the greatest common divisor of all the N that can appear in a decomposition of the diagonal of X.

Having a decomposition of the diagonal in Chow implies having one in cohomology. The converse is not true. But we already have produced a new interesting birational invariant:

Lemma 4. $N(X)$ *is a birational invariant of smooth projective varieties.*

Another interesting invariant, which is (surprisingly) closely related, is the Chow group $CH_0(X)$ of 0-cycles on X.

Lemma 5. $CH_0(X)$ *is a stable birational invariant of smooth projective varieties.*

The situation that is of interest to us in this talk is the case of trivial degree zero Chow group, i.e., $CH_0(X) = \mathbb{Z}$. Of course, if X is rationally connected, then $CH_0(X) = \mathbb{Z}$. But this should be true for a much larger class of varieties. The biggest conjecture in this direction is the following:

Conjecture 6 (Bloch)**.** *If* $H^0(X, \Omega_X^k) = 0$ *for* $k > 0$*, then* $CH_0(X) = \mathbb{Z}$*.*

On the other hand, having trivial CH_0 is itself a very strong property. By the following deep theorem, it has a lot to do with decompositions of the diagonal.

Theorem 7 (Bloch–Srinivas)**.** *If* $CH_0(X) = \mathbb{Z}$*, then* X *admits a Chow decomposition of the diagonal.*

Corollary 8. *All rationally connected varieties admit a decomposition of the diagonal.*

As a partial converse of the Bloch–Srinivas theorem, we get trivial CH_0 if our variety admits an *integral* decomposition of the diagonal.

Lemma 9. *If* X *admits an integral Chow decomposition of the diagonal, then* $CH_0(X) = \mathbb{Z}$*. In particular, all* $\mathbb{C}$*-points of* X *are rationally equivalent.*

Proof. Any cycle Γ in $X \times X$ acts on cycles in X by

$$\alpha \mapsto p_{1,*}(p_2^* \alpha \cdot \Gamma)$$

where $p_1, p_2 \colon X \times X \to X$ are the projections. Now the diagonal Δ_X acts as the identity and the right hand side $x \times X + Z$ of an integral Chow decomposition of the diagonal acts on a point $p \in X$ as

$$p_{1,*}\left((x \times X + Z) \cdot [X \times p]\right) = p_{1,*}\left((x \times X) \cdot (X \times p)\right) + p_{1,*}\left(Z \cdot [X \times p]\right)$$

Since $Z \subseteq X \times D$ we get $p_{1,*}(Z \cdot [X \times p]) = 0$. Furthermore, $p_{1,*}(x \times p) = x$, and hence for any $z \in CH_0(X)$ the action is given by $z \mapsto \deg(z) \cdot x$. Comparing both sides of the equation we get $z = \deg(z)x$ and $CH_0(X) = \mathbb{Z}$. □

As the previous results show, having a decomposition of the diagonal and having trivial CH_0 are not quite equivalent. The right property to ask for is *universal triviality* of CH_0.

Definition 10. We say that X has universally trivial CH_0 if for every field extension L of $\mathbb{C}$ we have $CH_0(X_L) = \mathbb{Z}$.

As we mentioned, if X is rationally connected, then $CH_0(X) = \mathbb{Z}$. But X might not have universally trivial CH_0. Next comes the proof of the equivalence that we promised:

Lemma 11. *A complex projective smooth variety X has universally trivial CH_0 if and only if X admits an integral Chow decomposition of the diagonal.*

Proof. Let $L/\mathbb{C}$ be a field extension. First assume that X admits an integral Chow decomposition of the diagonal with some $x \in X(\mathbb{C})$, hence $CH_0(X/\mathbb{C}) = \mathbb{Z}$. Then, by pulling back, X also admits a decomposition over L using the point x_L, hence $CH_0(X_L) = \mathbb{Z}$ as well.

Now assume that X has universally trivial CH_0. Take $L = \mathbb{C}(X)$, the function field of X. The generic point δ_L of X_L is in $CH_0(X_L) = CH^n(X_L)$. Its class in $CH^n(X_L)$ is the restriction of the diagonal class $\Delta_X \in CH^n(X \times X)$ to $X \times \operatorname{Spec} \mathbb{C}(X)$. Now fix some $x \in X(\mathbb{C})$. Since $CH_0(X_L) = \mathbb{Z}$, we know that δ_L has the same class as x_L. Hence $\Delta_X - [x \times X]$ vanishes when restricted to $CH^n(X_L)$. Therefore, for some Zariski open $U \subseteq X$ the class $\Delta_X - [x \times X]$ vanishes when restricted to $CH^n(X \times U)$. Let $D = X \setminus U$. By the localization exact sequence

$$CH^n(X \times D) \to CH^n(X \times X) \to CH^n(X \times U) \to 0$$

we may conclude that $\Delta_X - [x \times X]$ has the same class as some cycle Z supported on $X \times D$. $\square$

An easy consequence of the above is that all stably rational varieties admit integral Chow decompositions of the diagonal, since they have universally trivial CH_0.

There is also a result similar to Lemma 11 for non-integral decompositions:

Lemma 12. *X admits a decomposition of the diagonal if and only if*

$$A_0 = \{P \in CH_0(X) \mid \deg(P) = 0\}$$

is universally N-torsion for some integer N, meaning that for each field extension L of $\mathbb{C}$ we have $N \cdot A_0(X_L) = 0$. Moreover, $N(X)$ is the annihilator of the torsion.

As is suggested by the equivalence of universally trivial CH_0 and the existence of a decomposition of the diagonal, the latter is also a stable birational invariant:

Proposition 13. *The existence of an integral (Chow or cohomological) decomposition of the diagonal is a stable birational invariant of smooth projective varieties.*

After all this new theory, we want to see what this has to do with cubic fourfolds. First a small lemma connecting unirational parametrizations and decompositions of the diagonal.

Lemma 14. *If X admits a unirational parametrization of degree d, then $N(X) \mid d$.*

Proof. Let $n = \dim(X)$ and $\rho\colon \mathbb{P}^n \dashrightarrow X$ be a unirational parametrization of degree d. Take the integral decomposition of the diagonal of $\mathbb{P}^n$ in $\mathbb{P}^n \times \mathbb{P}^n$ and push it forward by $\rho \times \rho$, which gives $d \cdot \Delta_X = d \cdot [x \times X] + Z$. □

Theorem 15. *All cubic hypersurfaces of dimension at least two admit a unirational parametrization of degree* 2*.*

Proof. Let $X \subseteq \mathbb{P}^{n+1}$ be a cubic hypersurface. The proof depends only on the existence of a k-rational line L in X, so even if we work over some field k which is not algebraically closed, we still get the result if we suppose the existence of such a line. We then define

$$W = \{(p, l) \mid p \in L, l \text{ a line in } \mathbb{P}^{n+1} \text{ tangent to } X \text{ at } p\}$$

The projection map $W \to L$ exhibits W as a locally trivial $\mathbb{P}^{n-1}$-bundle. Consider a general line l through a point $p \in L$. Since X has degree 3 and l is tangent to X at p, the line meets X in one other point $g(p)$. This gives a rational map $g\colon W \dashrightarrow X$. We show g has degree 2. Let $q \in X$ be a general point. The plane through L and q intersects X in the union of L and a smooth conic Q. Since we assume generality, Q meets L in exactly two distinct points p_1 and p_2. Let l_1 and l_2 be the lines connecting p_1 and p_2 to q, respectively. Then $(p_1, l_1), (p_2, l_2) \in W$ are the two points mapping to q. □

Corollary 16. *Every cubic fourfold X admits a decomposition*

$$2\Delta_X = 2 \cdot [x \times X] + Z$$

For cubic hypersurfaces of certain dimensions we also have the following very convenient theorem which gets rid of the difference between Chow and cohomological decompositions. In practice, cohomology calculations are often much easier to handle, so that's good.

Theorem 17 ([Voi])**.** *Let X be a smooth cubic hypersurface of odd dimension or dimension four. Then X admits a Chow decomposition of the diagonal if and only if it admits a cohomological decomposition of the diagonal.*

Next up is the (up to now) central result on cubic fourfolds and decompositions of the diagonal:

Theorem 18 ([Voi], Theorem 5.6)**.** *Let X be a special cubic fourfold with discriminant $d \equiv 2 \pmod 4$. Then X admits an integral decomposition of the diagonal.*

Having an integral decomposition of the diagonal is implied by the existence of a unirational parametrization of odd degree, since we always have a parametrization of degree 2 and we can combine the decompositions obtained from coprime ones to yield an integral decomposition. Something similar is also used in the proof of the above theorem, although not for X. Other tools include the equivalence of the triviality of certain endomorphism groups

of Hodge structures and universal surjectivity of maps relating $CH_0(Y)$ and $CH_0(X)$ for $Y \subseteq X$.

Hence it is natural to ask whether the integral decomposition actually "comes from" the existence of unirational parametrizations.

Question 19. Does a special cubic fourfold of discriminant $d \equiv 2 \pmod 4$ admit a unirational parametrization of odd degree? Are these special cubic fourfolds stably rational?

Proposition 20 (Hassett)**.** *A generic cubic fourfold* $X \in \mathcal{C}_d$ *for* $d = 14, 18, 26, 30, 38$ *has a unirational parametrization of odd degree.*

The case of $d = 42$, answered by Lai, was explicitly asked for by Hassett in his survey (Example 43).

On the other hand, all cubic fourfolds satisfy a cohomological condition that is implied by the existence of an integral decomposition of the diagonal: they all have $H^3_{nr}(X_L, \mathbb{Q}/\mathbb{Z}) = 0$ for all field extensions L of $\mathbb{C}$ (this is a deep theorem by Voisin ([Voi15])). The same is true for all varieties with integral cohomological decompositions of the diagonal:

Theorem 21. *If* X *has an integral cohomological decomposition of the diagonal, then* $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}) = H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$.

Proof. As proved elsewhere, $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z})$ is the Artin–Mumford invariant $\operatorname{Tors} H^3(X, \mathbb{Z})$ and $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$ can be identified with the failure cycles of the integral Hodge conjecture $\operatorname{Hdg}^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{alg}$.

Now let

$$[\Delta_X] = [X \times x] + [Z] \in H^{2n}(X \times X, \mathbb{Z})$$

be an integral cohomological decomposition of the diagonal, where Z is supported on $D \times X$ with $D \subsetneq X$. In order to do calculations with the Hodge classes, we want to desingularize D so that we can pull back there. We choose D in such a way that for each component Z_i of the support of Z there is a divisorial component D_i of D containing Z_i and being generically smooth along Z_i. Furthermore, let $j\colon \widetilde{D} \to D \hookrightarrow X$ be a desingularization of D. Then Z lifts to a cycle $\widetilde{Z} \in CH^{n-1}(\widetilde{D} \times X)$.

We can rewrite the decomposition of the diagonal as

$$[\Delta_X] = [X \times x] + (j, \mathrm{id}_X)_*[\widetilde{Z}]$$

and let both sides act on classes in $H^*(X, \mathbb{Z})$. Assume $\alpha \in H^k(X, \mathbb{Z})$ with $k > 0$. The left side $[\Delta_X]$ acts as the identity, while $[X \times x]^*\alpha = 0$ because of degree reasons. Hence

$$\alpha = j_*([\widetilde{Z}]^*\alpha) \in H^k(X, \mathbb{Z}) \tag{1}$$

Now assume $\alpha \in \operatorname{Tors} H^3(X, \mathbb{Z})$. Then

$$[\widetilde{Z}]^*\alpha \in \operatorname{Tors} H^1(\widetilde{D}, \mathbb{Z}) = 0$$

because H^1 never has torsion (it is $\mathrm{Hom}(\pi_1(-), \mathbb{Z})$). Therefore $\alpha = 0$ by (1). Similarly, if $\alpha \in \mathrm{Hdg}^4(X, \mathbb{Z})$, then

$$[\widetilde{Z}]^*\alpha \in \mathrm{Hdg}^2(\widetilde{D}, \mathbb{Z}) = H^2(\widetilde{D}, \mathbb{Z})_{\mathrm{alg}}$$

by the Lefschetz $(1,1)$-theorem. Since the pushforward of algebraic classes is algebraic, we can conclude $\alpha \in H^4(X, \mathbb{Z})_{\mathrm{alg}}$ from (1). □

This motivates the next question:

Question 22. Is there a cubic fourfold X with $K_d = H^{2,2}(X, \mathbb{Z})$ where $d \equiv 0 \pmod 4$ and admitting an integral decomposition of the diagonal? Is there one admitting an odd unirational parametrization?

The real power of the technique of decomposing the diagonal stems from the ability to spread out such a cycle decomposition to "neighboring fibers" in a family.

Theorem 23 ([Voi15]). *Let $\pi\colon \mathcal{X} \to B$ be a flat projective morphism of relative dimension $n \geqslant 2$, where B is a smooth curve. Assume that the fiber $\mathcal{X}_t$ is smooth for $t \neq 0$, and has at worst ordinary quadratic singularities for $t = 0$. Then:*

1. *If for general $t \in B$ the fiber $\mathcal{X}_t$ admits a Chow decomposition of the diagonal, the same is true for any smooth projective model $\widetilde{\mathcal{X}}_0$ of $\mathcal{X}_0$.*

2. *If for general $t \in B$ the fiber $\mathcal{X}_t$ admits a cohomological decomposition of the diagonal, and the even degree integral homology of a smooth projective model $\widetilde{\mathcal{X}}_0$ of $\mathcal{X}_0$ is algebraic (i.e. generated over $\mathbb{Z}$ by classes of subvarieties), then $\widetilde{\mathcal{X}}_0$ also admits a cohomological decomposition of the diagonal.*

There are some improvements available on the singularity assumptions of the central fiber. The strategy to employ this theorem is as follows: find a singular central fiber that provably has no decomposition of the diagonal (or does not have universally trivial CH_0) and conclude from there that the general fiber does not admit such a decomposition. This means that the general fiber in $\mathcal{X} \to B$ cannot be stably rational. By patching together the one-dimensional families, you get results about the very general element in your moduli space. Voisin used this to prove the following:

Theorem 24 ([Voi15]). *The very general quartic double solid does not admit an integral cohomological decomposition of the diagonal. Similarly, the desingularization of the very general quartic double solid with $k \leqslant 7$ nodes in general position does not admit a cohomological decomposition of the diagonal.*

We end with a fairly recent theorem by Totaro that uses among other tools the above strategy.

Theorem 25 ([Tot16]). *A very general hypersurface of degree at least $2\lceil \frac{n+2}{3} \rceil$ is not stably rational.*

References

[Lai16] K.-W. Lai. (2016). New cubic fourfolds with odd degree unirational parametrizations. arXiv: `1606.03853` (cit. on p. 1).

[Tot16] B. Totaro, "Hypersurfaces that are not stably rational," *J. Amer. Math. Soc.*, vol. 29, pp. 883–891, 2016 (cit. on p. 6).

[Voi] C. Voisin, "On the universal CH_0 group of cubic hypersurfaces," *to appear in J. Eur. Math. Soc. (JEMS)*, (cit. on p. 4).

[Voi15] ——, "Unirational threefolds with no universal codimension 2 cycle," *Invent. Math.*, vol. 201, pp. 207–237, 2015 (cit. on pp. 5, 6).