Singularities of hypersurfaces and theta divisors

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These notes are completely based on the book [Laz04] and the course notes [Laz09] and contain no original thought whatsoever by myself.

1 Recap of results

Recall the following results.

Proposition 1.1 ([Laz04], Proposition 9.3.2)**.** *Assume that* X *has dimension* n*, and let* D *be an effective* $\mathbb{Q}$*-divisor on* X*. If* $\operatorname{mult}_x D \geqslant n$ *at some point* $x \in X$*, then* $\mathcal{J}(D)$ *is non-trivial at* x*, i.e.* $\mathcal{J}(D) \subseteq \mathfrak{m}_x$*, where* $\mathfrak{m}_x$ *is the maximal ideal of* x*. More generally, if*

$$\operatorname{mult}_x D \geqslant n + p - 1$$

for some integer $p \geqslant 1$*, then* $\mathcal{J}(D) \subseteq \mathfrak{m}_x^p$*.*

Proposition 1.2 ([Laz04], Example 9.3.5)**.** *Let* $Z \subseteq X$ *be an irreducible subvariety of codimension* e*, and let* D *be an effective* $\mathbb{Q}$*-divisor on* X*. If* $\operatorname{mult}_Z D \geqslant e$ *then*

$$\mathcal{J}(D) \subseteq \mathcal{I}_Z$$

Proof. Take a log resolution $\mu\colon X' \to X$ where the first step is blowing up X along Z. There is a unique component E of the exceptional locus of this blowup which maps surjectively onto Z. Then

$$\operatorname{mult}_E K_{X'/X} = e - 1$$

while

$$\operatorname{mult}_E \mu^* D = \operatorname{mult}_Z D =: \delta$$

so we have

$$\mathcal{J}(D) \subseteq \mu_* \mathcal{O}_{X'}((c - 1 - \delta)E)$$

and this is going to be contained in $\mathcal{I}_Z$ if $e - 1 - \delta \leqslant -1$ or, in other words, $\delta \geqslant e$. ■

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Proposition 1.3 ([Laz04], Proposition 9.4.26). *Let X be a smooth projective variety of dimension n, and fix a very ample divisor B on X. Let D be an effective $\mathbb{Q}$-divisor and L an integral divisor on X such that $L - D$ is big and nef. Then*

$$\mathcal{O}_X(K_X + L + mB) \otimes \mathcal{J}(D)$$

is globally generated as soon as $m \geqslant n$.

Proof. Mumford regularity combined with Nadel vanishing. ∎

2 Adjoint ideals

Recall that by the projection formula multiplier ideals of integral divisors carry no interesting information. There is a remedy for this.

Definition 2.1. Let X be a smooth variety and D a reduced integral divisor. Let $\mu\colon X' \to X$ be a log resolution of (X, D) and D' the proper transform of D. We may assume that D' is non-singular and write $\mu^*(D) = D' + F$ where F is the μ-exceptional integral divisor. The **adjoint ideal** of D in X is defined as

$$\operatorname{adj}(D) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - F)$$ ◊

Lemma 2.2. If $\nu\colon X'' \to X'$ is a birational morphism of smooth varieties, then

$$\nu_*(\mathcal{O}_{X''}(K_{X''})) = \mathcal{O}_{X'}(K_{X'})$$

Theorem 2.3 (Grauert–Riemenschneider vanishing, [Laz04] Theorem 4.3.9). *Let $f\colon X \to Y$ be a generically finite and surjective projective morphism of varietise, with X smooth. Then*

$$R^i f_*\mathcal{O}_X(K_X) = 0$$

for all $i > 0$.

Proposition 2.4. *Let (X, D) be as above and let $\nu\colon D' \to D$ be any resolution of singularities. Then we have the exact sequence*

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \otimes \operatorname{adj}(D) \to \nu_*\mathcal{O}_{D'}(K_{D'}) \to 0$$

Furthermore $\operatorname{adj}(D)$ is trivial if and only if D is normal with at worst rational singularities.

Proof. The sheaf $\nu_*\mathcal{O}_{D'}(K_{D'})$ is independent of the resolution chosen (reason: any two resolutions can be dominated by a third; then apply Lemma 2.2). We can therefore just work with a log resolution $\mu\colon X' \to X$ of (X, D) (as above), restricted to D'.

Consider the exact sequence

$$0 \to \mathcal{O}_{X'}(K_{X'}) \to \mathcal{O}_{X'}(K_{X'} + D') \to \mathcal{O}_{D'}(K_{D'}) \to 0$$

By Lemma 2.2 we have $\mu_*\mathcal{O}_{X'}(K_{X'}) = \mathcal{O}_X(K_X)$ and by the Grauert–Riemenschneider theorem we have $R^1\mu_*\mathcal{O}_{X'}(K_{X'}) = 0$. Furthermore

$$K_{X'} + D' = \mu^*(K_X + D) + (K_{X'/X} - F)$$

so the projection formula yields

$$\mu_*\mathcal{O}_{X'}(K_{X'} + D') = \mathcal{O}_X(K_X + D) \otimes \operatorname{adj}(D)$$

So pushing forward the exact sequence by μ gives exactly

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \otimes \operatorname{adj}(D) \to \mu_*\mathcal{O}_{D'}(K_{D'}) \to 0$$

Now observe that $\operatorname{adj}(D) = \mathcal{O}_X$ if and only if

$$\mu_*\mathcal{O}_{D'}(K_{D'}) = \mathcal{O}_D(K_X + D) = \omega_D$$

(compare to the ideal sheaf exact sequence of D). Since ω_D and $\mathcal{O}_{D'}(K_{D'})$ are line bundles, this cannot be true if D is not normal. A result of Kollár ([Kol]) then says the previous equality is true if and only if D has rational singularities (the Cohen–Macaulay hypotheses on D are satisfied). ■

3 Singularities of hypersurfaces

3.1 Conditions imposed on hypersurfaces by singularities

As a first application we will determine sufficient conditions for when the singularities of a given hypersurface impose independent conditions on other hypersurfaces. The classical case is the one of singular plane curves. We first consider only the set of cusps. The resulting bound goes back to Zariski.

Proposition 3.1 (Zariski). *Let $C \subseteq \mathbb{P}^2$ be a reduced curve of degree m and Ξ the reduced set of cusps of C. Then*

$$H^1(\mathbb{P}^2, \mathcal{I}_\Xi(k)) = 0 \quad \textit{for all } k > \frac{5}{6}m - 3$$

Proof. We can determine the log canonical threshold of a cusp to be $\frac{5}{6}$ (see Example 9.3.13 in [Laz04]), so it is clear where that number comes from. Set $D = \frac{5}{6}C$. We then have

$$0 \to \mathcal{J}(D) \to \mathcal{I}_\Xi \to \mathcal{I}_\Xi/\mathcal{J}(D) \to 0$$

where the quotient is supported at only finitely many points (since C is reduced and so $\mathcal{J}(D)$ is nontrivial only at isolated points). Consequently

$$0 \to \mathcal{J}(D) \otimes \mathcal{O}_{\mathbb{P}^2}(l) \to \mathcal{I}_\Xi \otimes \mathcal{O}_{\mathbb{P}^2}(l) \to \mathcal{I}_\Xi/\mathcal{J}(D) \otimes \mathcal{O}_{\mathbb{P}^2}(l) \to 0$$

and the map

$$H^1(\mathbb{P}^2, \mathcal{J}(D) \otimes \mathcal{O}_{\mathbb{P}^2}(l)) \to H^1(\mathbb{P}^2, \mathcal{I}_\Xi(l)) \to 0$$

is surjective. We can now apply Nadel vanishing with $L = \mathcal{O}_{\mathbb{P}^2}(l)$ as soon as $l > \frac{5}{6}m$ and hence

$$H^1(\mathbb{P}^2, \mathcal{J}(D) \otimes \mathcal{O}_{\mathbb{P}^2}(l - 3)) = 0$$

■

We can not do as well when we allow the full set of singularities of C.

Proposition 3.2. *Let $C \subseteq \mathbb{P}^2$ be a reduced curve of degree m and let Σ be the reduced set of singular points of C. Then Σ imposes independent conditions on curves of degree $k \geqslant m-2$, i.e.*

$$H^1(\mathbb{P}^2, \mathcal{I}_\Sigma(k)) = 0 \quad \text{for all } k \geqslant m-2$$

Remark 3.3. Consider the reducible curve C given by four general lines in $\mathbb{P}^2$, meeting transversely in six double points. Then C has degree 4 and

$$0 \to \mathcal{I}_\Sigma(1) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_\Sigma(1) \to 0$$

allows us to calculate

$$\chi(\mathcal{I}_\Sigma(1)) = \chi(\mathcal{O}_{\mathbb{P}^2}(1)) - 6 = -3$$

while $H^0(\mathbb{P}^2, \mathcal{I}_\Sigma(1)) = 0$ since no line passes through all six points. This shows

$$\dim H^1(\mathbb{P}^2, \mathcal{I}_\Sigma(1)) = 3$$

and the singularities of C do not impose independent conditions on lines, as can also be seen from the picture. So the bound given in 3.2 is sharp in the general case. We can improve the bound by 1 in the case of irreducible curves, though. $\diamondsuit$

Proof of Proposition 3.2. Let Γ be a reduced curve of some degree l that passes through Σ and does not contain any component of C. Choose some $0 < \varepsilon \ll 1$ and set

$$D = (1-\varepsilon)C + 2\varepsilon\Gamma$$

Since $\operatorname{mult}_x D \geqslant 2$ for all $x \in \Sigma$, we have $\mathcal{J}(D) \subseteq \mathcal{I}_\Sigma$ and D is reduced, hence the quotient $\mathcal{I}_\Sigma/\mathcal{J}(D)$ is supported on a finite set as before. We again have

$$H^1(\mathbb{P}^2, \mathcal{J}(D) \otimes \mathcal{O}_{\mathbb{P}^2}(k)) \to H^1(\mathbb{P}^2, \mathcal{I}_\Xi(k)) \to 0$$

surjective. Since $\varepsilon \ll 1$,

$$\deg D = (1-\varepsilon)m + 2\varepsilon l < m+1$$

and therefore Nadel vanishing with some L of degree at least $m+1$ applies. This is equivalent to $k \geqslant m+1-3 = m-2$. ∎

Proposition 3.4. *Let $C \subseteq \mathbb{P}^2$ be an* integral *curve of degree m and Σ the set of singular points of C. Then Σ poses independent conditions on curves of degree $k \geqslant m-3$, i.e.*

$$H^1(\mathbb{P}^2, \mathcal{I}_\Sigma(k)) = 0 \quad \text{for all } k \geqslant m-3$$

Proof. We only need to show the statement for $k = m-3$. Let $\nu\colon C' \to C$ be the normalization of C. The adjoint ideal of C is contained in $\mathcal{I}_\Sigma$, hence as in the previous proof

$$H^1(\mathbb{P}^2, \operatorname{adj}(C) \otimes \mathcal{O}_{\mathbb{P}^2}(m-3)) \to H^1(\mathbb{P}^2, \mathcal{I}_\Sigma(m-3)) \to 0$$

is surjective. Recall the short exact sequence of the adjoint ideal in 2.4 where we set $D = C$ and hence $\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(-3)$ and $\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2} + D) = \mathcal{O}_{\mathbb{P}^2}(m-3)$:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(m-3) \otimes \operatorname{adj}(C) \to \nu_*\mathcal{O}_{C'}(K_{C'}) \to 0$$

Since $H^1(\mathcal{O}_{\mathbb{P}^2}(-3)) = 0$ we get the following exact sequence in cohomology:

$$0 \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-3) \otimes \operatorname{adj}(C)) \to H^1(C', \omega_{C'}) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \to 0$$

Since the last two spaces are 1-dimensional, we must have $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-3) \otimes \operatorname{adj}(C)) = 0$ and therefore $H^1(\mathbb{P}^2, \mathcal{I}_\Sigma(m-3)) = 0$ as well. ■

The above propositions were generalized by Severi to the case of hypersurfaces in $\mathbb{P}^3$ and by Park and Woo ([PW06]) to hypersurfaces in arbitrary projective space $\mathbb{P}^r$.

Proposition 3.5 (Park, Woo [PW06])**.** *Let $S \subseteq \mathbb{P}^r$ be a hypersurface of degree m with only isolated singularities and Σ be the reduced set of singular points. Then Σ imposes independent conditions on hypersurfaces of degree $k \geqslant m(r-1) - (2r-1)$, i.e.*

$$H^1(\mathbb{P}^r, \mathcal{I}_\Sigma(k)) = 0 \quad \text{for all } k \geqslant m(r-1) - (2r-1)$$

3.2 Minimal degree of hypersurfaces containing given points

Our next application is as follows. Suppose we are given a finite set S of points in projective space. What is the minimal degree of a hypersurface containing them? This question might be particularly interesting when $k \neq \overline{k}$, i.e. when looking for rational points.

In some applications we may know there exists a hypersurface of high degree through S, but with high multiplicity at each point $x \in S$. It turns out that in this case we can "decrease" the degree of the containing hypersurface, in a sense flattening it by decreasing the multiplicities at the given points.

Proposition 3.6. *Let $S \subseteq \mathbb{P}^n$ be a finite set of points and let X be a hypersurface of degree d such that*

$$\operatorname{mult}_x X \geqslant k \quad \text{for all } x \in S$$

for some positive integer k. Then S is contained in a hypersurface of degree $\leqslant \lfloor \frac{dn}{k} \rfloor$.

Remark 3.7. The given bound seems to be non-optimal. Chudnovsky conjectures that we can actually have a hypersurface A with

$$\deg A \leqslant \left\lfloor \frac{dn}{k} \right\rfloor - (n-1)$$

This is so far only known in the case of plane curves. ♢

Proof of Proposition 3.6. Let $\delta = \lfloor dn/k \rfloor$. We have to show that $H^0(\mathbb{P}^n, \mathcal{I}_S(l)) \neq 0$ as soon as $l < \delta$. Let $D = \frac{n}{k}X$. Then

$$\operatorname{mult}_x D \geqslant n \quad \text{for all } x \in S$$

and hence $\mathcal{J}(D) \subseteq \mathcal{I}_S$ by Proposition 1.1. Hence

$$0 \to H^0(\mathbb{P}^n, \mathcal{J}(D) \otimes \mathcal{O}_{\mathbb{P}^n}(l)) \hookrightarrow H^0(\mathbb{P}^n, \mathcal{I}_S(l))$$

If H is a hyperplane divisor of $\mathbb{P}^n$ then

$$lH - D \equiv_{\text{num}} \left(l - \frac{dn}{k}\right) H$$

Now Proposition 1.3 implies that

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-(n+1)H + lH + nH) \neq 0$$

if $lH - D$ is big and nef, which is the case if we let $l \geqslant \delta + 1$. Then

$$[-(n+1) + l + n]H = (l-1)H = \delta H$$

hence S is contained in a hypersurface of degree at most δ. ■

In fact this theorem is just a special case of a theorem by Skoda:

Theorem 3.8 (Skoda, see [EV83]). *Let $S \subseteq \mathbb{P}^n$ be a finite set of points. For any $t \geqslant 1$ let*

$$\omega_t(S) = \min\{\deg X \mid X \textit{ hypersurface in } \mathbb{P}^n, \operatorname{mult}_x X \geqslant t \textit{ for all } x \in S\}$$

Then for $t' < t$

$$\frac{\omega_{t'}(S)}{t' + n - 1} \leqslant \frac{\omega_t(S)}{t}$$

Proof. Let X be a hypersurface of degree $d = \omega_t(S)$ with $\operatorname{mult}_x X \geqslant t$ for all $x \in S$. Set

$$\delta = \frac{t' + n - 1}{t}, \quad D = \delta X$$

Then $\operatorname{mult}_x D \geqslant t' + n - 1$ along S and by Proposition 1.1 we have $\mathcal{J}(D) \subseteq \mathcal{I}_S^{t'}$. Now argue along the proof of the previous proposition to find

$$H^0(\mathbb{P}^n, \mathcal{I}_S^{t'}(d\delta)) \neq 0$$

i.e. there is a hypersurface of degree $d\delta$ vanishing at S with multiplicity at least t'. This implies

$$\omega_{t'}(S) \leqslant d\delta = (t' + n - 1)\frac{\omega_t(S)}{t}$$ ■

Remark 3.9. Using Proposition 1.2 we can generalize to sets S of arbitrary codimension, not just points. Let $S \subseteq \mathbb{P}^n$ have codimension at most e. If X is a hypersurface of degree d passing through S such that

$$\operatorname{mult}_x X \geqslant k \quad \text{for all } x \in S$$

then S lies on a hypersurface of degree $\leqslant \lfloor \frac{de}{k} \rfloor$. The proof is completely analogous. ◊

4 Singularities of theta divisors

Consider a smooth curve C of genus g. Its Jacobian $J(C) \cong \mathrm{Pic}^{g-1}(C)$ is a principally polarized abelian variety of dimension g and the theta divisor can be described (as a set) as the locus of degree $g-1$ line bundles that admit a section. We also have a description of the singular locus of Θ. For a general reference, see [Arb+85].

Theorem 4.1 (Riemann's singularity theorem)**.** *For every effective divisor D of degree $g-1$ on C we have*

$$\mathrm{mult}_{\mathcal{O}(D)}(\Theta) = h^0(C, \mathcal{O}_C(D))$$

The proof is classical and uses derivatives of Riemann's theta function and similar stuff. By elementary theorems on the dimension of the spaces of linear series on C we get the following

Corollary 4.2. If C is hyperelliptic, then

$$\dim \mathrm{Sing}(\Theta) = g - 4$$

Otherwise

$$\dim \mathrm{Sing}(\Theta) = g - 3$$

For $g \geqslant 4$ we know that the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ sending a curve to its Jacobian embeds $\mathcal{M}_g$ as a proper subvariety of $\mathcal{A}_g$. It is a long-standing question to give characterizations of its image. In other words, how can we distinguish Jacobian varieties from the general abelian variety of dimension g? This is known as the **Schottky problem**.

Some decades ago, Andreotti and Mayer tried a characterization in terms of the dimension of the singular locus of the theta divisor. We define the **Andreotti–Mayer loci** to be

$$N_k = \{A \in \mathcal{A}_g \mid \dim \mathrm{Sing}(\Theta_A) \geqslant k\}$$

Then there is the following theorem:

Theorem 4.3 (Andreotti–Mayer, [AM67])**.** *There is a unique irreducible component of N_{g-4} that contains the image of $\mathcal{M}_g$.*

The proof uses the heat equation for Riemann's theta function. However, this theorem is not enough for a complete characterization. Already for $g = 4$ there is an extra component θ_{null} consisting of all ppav where the singular locus of Θ contains a two-torsion point and this phenomenon just becomes worse in higher dimensions (see e.g. [Bea77]).

Going back to the Riemann singularity theorem, we might ask for another generalization: How bad can the singularities of Θ on *any* fixed ppav be? For instance, if Θ is irreducible, is it normal? Several answers were conjectured but only after the proof of the Kawamata–Viehweg vanishing theorem did Kollár realize that one could use it to say something about theta divisors on abelian varieties. In the following we let (A, Θ) be a principally polarized abelian variety of dimension g.

Theorem 4.4 ([Laz04], Theorem 10.1.6)**.** *The pair* (A, Θ) *is log-canonical, i.e. for all* $\varepsilon \in (0,1)$ *we have*

$$\mathcal{J}(A, (1-\varepsilon)\Theta) = \mathcal{O}_A$$

In particular, the codimension of

$$\Sigma_k = \{x \in A \mid \operatorname{mult}_x \Theta \geqslant k\}$$

is at least k*. This also means that no point in* Θ *can have multiplicity higher than* g*.*

Proof. Suppose to the contrary that there is an $\varepsilon \in (0,1)$ such that $\mathcal{J} = \mathcal{J}(A, (1-\varepsilon)\Theta)$ is not trivial. Let Z be the subscheme associated to $\mathcal{J}$. Observe that $Z \subseteq \Theta$. Now consider the exact sequence

$$0 \to \mathcal{J} \otimes \mathcal{O}_A(\Theta) \to \mathcal{O}_A(\Theta) \to \mathcal{O}_Z(\Theta) \to 0$$

The Nadel vanishing theorem (with $L = \Theta, D = (1-\varepsilon)\Theta, K_A = 0$) tells us $H^1(A, \mathcal{J} \otimes \mathcal{O}_A(\Theta)) = 0$, so the long exact sequence in cohomology goes like this:

$$0 \to H^0(\mathcal{O}_A(\Theta) \otimes \mathcal{J}) \to H^0(\mathcal{O}_A(\Theta)) \xrightarrow{\alpha} H^0(\mathcal{O}_Z(\Theta)) \to 0$$

Now Θ has exactly one global section, vanishing precisely on Θ, hence also on Z. So α must be the zero map, but at the same time surjective. Hence $H^0(\mathcal{O}_Z(\Theta)) = 0$.

Denote by $\Theta_a = \Theta + a$ the translate of Θ. For general a, Z is not contained in Θ_a, hence $H^0(Z, \mathcal{O}_Z(\Theta_a)) \neq 0$. Putting all Θ_a in a family and letting $a \to 0$ we see $H^0(Z, \mathcal{O}_Z(\Theta)) \neq 0$ by semicontinuity. This is a contradiction.

To get the last statement fix some k, take $\varepsilon \ll 1$ and apply Proposition 1.2 in the case $p = 1$ to the subvariety Σ_k. Since $\mathcal{J}((1-\varepsilon)\Theta) = \mathcal{O}_A$, we must have

$$(1-\varepsilon)k = \operatorname{mult}_{\Sigma_k}(1-\varepsilon)\Theta < \operatorname{codim}_A \Sigma_k$$

and hence $\operatorname{codim}_A \Sigma_k \geqslant k$. ■

Remark 4.5. Let $A = E_1 \times E_2$ be the product of two elliptic curves. Then

$$\Theta_A = \Theta_{E_1} \times E_2 \cup E_1 \times \Theta_{E_2} = \{0_{E_1}\} \times E_2 \cup E_1 \times \{0_{E_2}\}$$

and the two parts meet in $\{0_{E_1}\} \times \{0_{E_2}\}$, which has codimension 2. Hence $\operatorname{codim}_A \Sigma_2 = 2$, which is the maximum allowed by the theorem.

The following theorem shows that this is the only way in which we can have equality. If A doesn't split appropriately, the inequality on the codimension is strict. ◊

To say something about the type of singularities of Θ we need some intermediary results. They are of independent interest.

Theorem 4.6 (Green–Lazarsfeld's generic vanishing theorem)**.** *Let* X *be a smooth projective variety of dimension* n*. If* X *has maximal Albanese dimension, then*

$$H^i(X, P) = 0 \textit{ for all } j < n \textit{ and generic } P \in \operatorname{Pic}^0(X)$$

If $X \to A$ *is a generically finite mapping to an abelian variety, then the vanishing holds for the pullback to* X *of a generic element of* $\operatorname{Pic}^0(A)$*.*

Theorem 4.7 (Ueno, see [Mor87] Theorem 3.7). *Let X be an irreducible reduced subvariety of an abelian variety A. Let B be the connected component of the identity of $\{a \in A \mid a+X \subseteq X\}$. Then*

1. $X \to Y = X/B$ *is an étale fiber bundle with fiber* B *and* $X \times_Y Y' = Y' \times B$ *for some finite étale covering* Y' *of* Y.
2. $X \to Y$ *is (birational to) the Iitaka fibering of* X.
3. Y *is of general type.*

Corollary 4.8. The resolution of singularities of the theta divisor of a ppav is a variety of general type.

Theorem 4.9 (Kawamata–Viehweg [KV80]). *Let A be an abelian variety of dimension g and $X \to D$ a resolution of singularities of a divisor in A such that X is of general type. If $p_g(X) \leqslant g$ then $\chi(\mathcal{O}_X) \neq 0$.*

Now we can state and prove the following theorem of Ein and Lazarsfeld, showing that Θ_A can have at worst rational singularities. In particular, the question about normality of an irreducible theta divisor is settled.

Theorem 4.10 (Ein–Lazarsfeld [EL97]). *Let (A, Θ) be a ppav and assume that Θ is irreducible. Then Θ is normal and has only rational singularities. Consequently, Σ_k has a component of codimension exactly k if and only if (A, Θ) splits as a k-fold product of ppavs.*

Proof. We will show that Θ has at worst rational singularities. The statement about the splitting then follows by induction. In light of Proposition 2.4 we only need to prove $\mathrm{adj}(\Theta) = \mathcal{O}_A$.

Take a resolution of singularities $\nu \colon X \to \Theta$ and write down the exact sequence for the adjoint ideal from Proposition 2.4:

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(\Theta) \otimes \mathrm{adj}(\Theta) \to \nu_*(\mathcal{O}_X(K_X)) \to 0 \tag{4.1}$$

Take a generic line bundle $P \in \mathrm{Pic}^0(A)$ and twist this sequence to get

$$0 \to P \to \mathcal{O}_A(\Theta) \otimes P \otimes \mathrm{adj}(\Theta) \to \nu_*(\mathcal{O}_X(K_X)) \otimes P \to 0 \tag{4.2}$$

Observe that by the universal property of the Albanese variety, A is the Albanese of X and we can apply the generic vanishing theorem 4.6 together with Serre duality to obtain

$$H^i(X, \mathcal{O}_X(K_X + \nu^* P)) = H^{n-i}(X, \nu^*(P^\vee)) = 0$$

for all $i > 0$. It follows

$$H^0(X, P \otimes \nu_* \mathcal{O}_X(K_X)) = \chi(X, \mathcal{O}_X(K_X) \otimes \nu^* P) = \chi(X, \mathcal{O}_X(K_X)) = \chi(X, \mathcal{O}_X)$$

From (4.1) we get

$$H^0(A, \mathcal{O}_A) \cong H^0(\mathcal{O}_A(\Theta) \otimes \mathrm{adj}(\Theta))$$

and then

$$0 \to H^0(X, \mathcal{O}_X(K_X)) \to H^1(A, \mathcal{O}_A)$$

so $p_g(X) = h^0(X, K_X) \leqslant g$. We can now apply Corollary 4.8 and 4.9 to see $\chi(X, \mathcal{O}_X) \neq 0$ which in turn implies

$$H^0(X, P \otimes \nu_* \mathcal{O}_X(K_X)) \neq 0$$

Because $H^i(A, P) = 0$ for all i, it follows from (4.2) that

$$H^0(A, \mathcal{O}_A(\Theta) \otimes P \otimes \mathrm{adj}(\Theta)) \neq 0$$

and hence for a general translation $\mathcal{O}_A(\Theta + a)$

$$H^0(A, \mathcal{O}_A(\Theta + a) \otimes \mathrm{adj}(\Theta)) \neq 0 \tag{4.3}$$

Suppose $\mathrm{adj}(A) \neq \mathcal{O}_A$ and let Z be the zero scheme of $\mathrm{adj}(A)$. Equation (4.3) then says that for a general a we have $Z \subseteq (\Theta + a)$. This is absurd. ■

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